

APM462 Fall 2019

Lecture Notes

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1 Matrix Calculus

Row v.s. Column Vector Our default rule is that every vector is a column vector unless explicitly stated otherwise.

This is also known as the numerator layout.

Special case: For $f : \mathbb{R}^n \rightarrow \mathbb{R}$, Df is a $1 \times n$ matrix or row vector.

1.1 Matrix Multiplication

Definition 1.1.1 Let A be $m \times n$, and B be $n \times p$, and let the product AB be

$$C = AB$$

then C is a $m \times p$ matrix, with element (i, j) given by

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

for all $i = 1, 2, \dots, m, j = 1, 2, \dots, p$.

Proposition 1.1.2 Let A be $m \times n$, and x be $n \times 1$, then the typical element of the product

$$z = Ax$$

is given by

$$z_i = \sum_{k=1}^n a_{ik} x_k$$

for all $i = 1, 2, \dots, m$.

Similarly, let y be $m \times 1$, then the typical element of the product

$$z^T = y^T A$$

is given by

$$z_i^T = \sum_{k=1}^n a_{ki} y_k$$

for all $i = 1, 2, \dots, n$.

Finally, the scalar resulting from the product

$$\alpha = y^T Ax$$

is given by

$$\alpha = \sum_{j=1}^m \sum_{k=1}^n a_{jk} y_j x_k$$

1.2 Partitioned Matrices

Proposition 1.2.1 Let A be a square, nonsingular matrix of order m . Partition A as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

so that A_{11} and A_{22} are invertible.

Then

$$A^{-1} = \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix}$$

proof:

Direct multiplication of the proposed A^{-1} and A yields

$$A^{-1}A = I$$

■

1.3 Matrix Differentiation

Proposition 1.3.1

$$\frac{\partial A}{\partial x} = \frac{\partial A^T}{\partial x}$$

Proposition 1.3.2

Let

$$y = Ax$$

where y is $m \times 1$, x is $n \times 1$, A is $m \times n$, and A does not depend on x . Suppose that x is a function of the vector z , while A is independent of z . Then

$$\frac{\partial y}{\partial z} = A \frac{\partial x}{\partial z}$$

Proposition 1.3.3

Let the scalar α be defined by

$$\alpha = y^T Ax$$

where y is $m \times 1$, x is $n \times 1$, A is $m \times n$, and A is independent of x and y , then

$$\frac{\partial \alpha}{\partial x} = y^T A$$

and

$$\frac{\partial \alpha}{\partial y} = x^T A^T$$

Proposition 1.3.4

For the special case where the scalar α is given by the quadratic form

$$\alpha = x^T Ax$$

where x is $n \times 1$, A is $n \times n$, and A does not depend on x , then

$$\frac{\partial \alpha}{\partial x} = x^T (A + A^T)$$

proof:

By definition

$$\alpha = \sum_{j=1}^n \sum_{i=1}^n a_{ij} x_i x_j$$

Differentiating with respect to the k th element of x we have

$$\frac{\partial \alpha}{\partial x_k} = \sum_{j=1}^n a_{kj} x_j + \sum_{i=1}^n a_{ik} x_i$$

for all $k = 1, 2, \dots, n$, and consequently,

$$\frac{\partial \alpha}{\partial x} = x^T A^T + x^T A = x^T (A^T + A)$$

■

Proposition 1.3.4

For the special case where A is a symmetric matrix and

$$\alpha = x^T Ax$$

where x is $n \times 1$, A is $n \times n$, and A does not depend on x , then

$$\frac{\partial \alpha}{\partial x} = 2x^T A$$

Proposition 1.3.5 Let the scalar α be defined by

$$\alpha = y^T x$$

where y is $n \times 1$, x is $n \times 1$, and both y and x are functions of the vector z . Then

$$\frac{\partial \alpha}{\partial z} = x^T \frac{\partial y}{\partial z} + y^T \frac{\partial x}{\partial z}$$

Proposition 1.3.6 Let the scalar α be defined by

$$\alpha = x^T x$$

where x is $n \times 1$, and x is a functions of the vector z . Then

$$\frac{\partial \alpha}{\partial z} = 2x^T \frac{\partial x}{\partial z}$$

Proposition 1.3.7 Let the scalar α be defined by

$$\alpha = y^T A x$$

where y is $m \times 1$, A is $m \times n$, x is $n \times 1$, and both y and x are functions of the vector z , while A does not depend on z . Then

$$\frac{\partial \alpha}{\partial z} = x^T A^T \frac{\partial y}{\partial z} + y^T A \frac{\partial x}{\partial z}$$

Proposition 1.3.8 Let A be an invertible, $m \times m$ matrix whose elements are functions of the scalar parameter α . Then

$$\frac{\partial A^{-1}}{\partial \alpha} = -A^{-1} \frac{\partial A}{\partial \alpha} A^{-1}$$

proof:

Start with the definition of the inverse

$$A^{-1} A = I$$

and differentiate, yielding

$$A^{-1} \frac{\partial A}{\partial \alpha} + \frac{\partial A^{-1}}{\partial \alpha} A = 0$$

rearranging the terms yields

$$\frac{\partial A^{-1}}{\partial \alpha} = -A^{-1} \frac{\partial A}{\partial \alpha} A^{-1}$$

■

Vector-by-vector Differentiation Identities 1.3.9

Young's Theorem 1.3.10 i.e. Symmetry of second derivatives

$$[\nabla_{xy} f(x, y)]^T = \nabla_{yx} f(x, y)$$

proof:

This is straightforward by writing out the elements of the matrix.

■

2 Second-year Calculus Review

functions $\mathbb{R} \rightarrow \mathbb{R}$

Condition	Expression	Numerator layout, i.e. by \mathbf{y} and \mathbf{x}^\top	Denominator layout, i.e. by \mathbf{y}^\top and \mathbf{x}
\mathbf{a} is not a function of \mathbf{x}	$\frac{\partial \mathbf{a}}{\partial \mathbf{x}} =$	$\mathbf{0}$	
	$\frac{\partial \mathbf{x}}{\partial \mathbf{x}} =$	\mathbf{I}	
\mathbf{A} is not a function of \mathbf{x}	$\frac{\partial \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} =$	\mathbf{A}	\mathbf{A}^\top
\mathbf{A} is not a function of \mathbf{x}	$\frac{\partial \mathbf{x}^\top \mathbf{A}}{\partial \mathbf{x}} =$	\mathbf{A}^\top	\mathbf{A}
a is not a function of \mathbf{x} , $\mathbf{u} = \mathbf{u}(\mathbf{x})$	$\frac{\partial a \mathbf{u}}{\partial \mathbf{x}} =$	$a \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$	
$v = v(\mathbf{x}), \mathbf{u} = \mathbf{u}(\mathbf{x})$	$\frac{\partial v \mathbf{u}}{\partial \mathbf{x}} =$	$v \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \mathbf{u} \frac{\partial v}{\partial \mathbf{x}}$	$v \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial v}{\partial \mathbf{x}} \mathbf{u}^\top$
\mathbf{A} is not a function of \mathbf{x} , $\mathbf{u} = \mathbf{u}(\mathbf{x})$	$\frac{\partial \mathbf{A} \mathbf{u}}{\partial \mathbf{x}} =$	$\mathbf{A} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$	$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{A}^\top$
$\mathbf{u} = \mathbf{u}(\mathbf{x}), \mathbf{v} = \mathbf{v}(\mathbf{x})$	$\frac{\partial (\mathbf{u} + \mathbf{v})}{\partial \mathbf{x}} =$	$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}$	
$\mathbf{u} = \mathbf{u}(\mathbf{x})$	$\frac{\partial \mathbf{g}(\mathbf{u})}{\partial \mathbf{x}} =$	$\frac{\partial \mathbf{g}(\mathbf{u})}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$	$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \frac{\partial \mathbf{g}(\mathbf{u})}{\partial \mathbf{u}}$
$\mathbf{u} = \mathbf{u}(\mathbf{x})$	$\frac{\partial \mathbf{f}(\mathbf{g}(\mathbf{u}))}{\partial \mathbf{x}} =$	$\frac{\partial \mathbf{f}(\mathbf{g})}{\partial \mathbf{g}} \frac{\partial \mathbf{g}(\mathbf{u})}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$	$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \frac{\partial \mathbf{g}(\mathbf{u})}{\partial \mathbf{u}} \frac{\partial \mathbf{f}(\mathbf{g})}{\partial \mathbf{g}}$

2.1 Mean Value Theorem in 1 Dimension

$g \in C^1$ on \mathbb{R}

$$\frac{g(x+h) - g(x)}{h} = g'(x + \theta h)$$

where $\theta \in (0, 1)$

Or equivalently,

$$g(x+h) = g(x) + hg'(x + \theta h)$$

2.2 1st Order Taylor Approximation

$g \in C^1$ on \mathbb{R}

$$g(x+h) = g(x) + hg'(x) + o(h)$$

where $o(h)$ is “little o ” of h , the error term.

Say a function $f(h) = o(h)$, this means $\lim_{h \rightarrow 0} \frac{f(h)}{h} = 0$

For example, for $f(h) = h^2$, we can say $f(h) = o(h)$,

since $\lim_{h \rightarrow 0} \frac{f(h)}{h} = \lim_{h \rightarrow 0} \frac{h^2}{h} = \lim_{h \rightarrow 0} h = 0$

proof: (Use MVT):

WTS : $g(x+h) - g(x) - hg'(x) = o(h)$

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{[g(x+h) - g(x)] - hg'(x)}{h} &= \lim_{h \rightarrow 0} \frac{[hg'(x + \theta h)] - hg'(x)}{h} \\
 &= \lim_{h \rightarrow 0} g'(x + \theta h) - g'(x) \\
 &= \lim_{h \rightarrow 0} g'(x) - g'(x) \\
 &= 0
 \end{aligned}$$



2.3 2nd Order Mean Value Theorem

$g \in C^2$ on \mathbb{R}

$$g(x+h) = g(x) + hg'(x) + \frac{h^2}{2}g''(x+\theta h)$$

for some $\theta \in (0,1)$

proof:

$$\text{WTS: } g(x+h) - g(x) - hg'(x) - \frac{h^2}{2}g''(x) = o(h^2)$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{g(x+h) - g(x) - hg'(x) - \frac{h^2}{2}g''(x)}{h^2} &= \lim_{h \rightarrow 0} \frac{[\frac{h^2}{2}g''(x+\theta h)] - \frac{h^2}{2}g''(x)}{h^2} \\ &= \lim_{h \rightarrow 0} \frac{1}{2}(g''(x+\theta h) - g''(x)) \\ &= \lim_{h \rightarrow 0} \frac{1}{2}(g''(x) - g''(x)) \\ &= 0 \end{aligned}$$



multivariate functions: $\mathbb{R}^n \rightarrow \mathbb{R}$

2.4 Recall: Definition of gradient

Gradient of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at $x \in \mathbb{R}^n$ (denoted $\nabla f(x)$) if exists is a vector characterized by the property:

$$\lim_{\mathbf{v} \rightarrow \mathbf{0}} \frac{f(\mathbf{x} + \mathbf{v}) - f(\mathbf{x}) - \nabla f(\mathbf{x}) \cdot \mathbf{v}}{\|\mathbf{v}\|} = 0$$

In Cartesian coordinates, $\nabla f(\mathbf{x}) = (\frac{\partial f}{\partial x_1}(\mathbf{x}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{x}))$

2.5 Mean Value Theorem in n dimension

$f \in C^1$ on \mathbb{R}^n , then for any $\mathbf{x}, \mathbf{v} \in \mathbb{R}^n$,

$$f(\mathbf{x} + \mathbf{v}) = f(\mathbf{x}) + \nabla f(\mathbf{x} + \theta \mathbf{v}) \cdot \mathbf{v}$$

for some $\theta \in (0,1)$

proof: Reduce to 1-dimension case

$$g(t) := f(\mathbf{x} + t\mathbf{v}), t \in \mathbb{R}$$

$$\begin{aligned} g'(t) &= \frac{d}{dt} f(\mathbf{x} + t\mathbf{v}) \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{v}) \cdot \frac{d(\mathbf{x} + t\mathbf{v})_i}{dt} && \text{(by Chain Rule)} \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{v}) \cdot \frac{d(\mathbf{x}_i + t\mathbf{v}_i)}{dt} \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{v}) \cdot \mathbf{v}_i \\ &= \nabla f(\mathbf{x} + t\mathbf{v}) \cdot \mathbf{v} && (*) \end{aligned}$$

$g \in C^1$ on \mathbb{R}

Using MVT in \mathbb{R} :

$$\begin{aligned}
 f(\mathbf{x} + \mathbf{v}) &= g(1) \\
 &= g(0 + 1) \\
 &= g(0) + 1g'(0 + \theta 1) & (\theta \in (0, 1)) \\
 &= g(0) + g'(\theta) \\
 &= f(\mathbf{x}) + \nabla f(\mathbf{x} + \theta \mathbf{v}) \cdot \mathbf{v} & (\text{by } (*))
 \end{aligned}$$

■

2.6 1st Order Taylor Approximation in \mathbb{R}^n

$f \in C^1$ on \mathbb{R}^n

$$f(\mathbf{x} + \mathbf{v}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{v} + o(\|\mathbf{v}\|)$$

proof:

$$\begin{aligned}
 \lim_{\|\mathbf{v}\| \rightarrow 0} \frac{[f(\mathbf{x} + \mathbf{v}) - f(\mathbf{x})] - \nabla f(\mathbf{x}) \cdot \mathbf{v}}{\|\mathbf{v}\|} &= \lim_{\|\mathbf{v}\| \rightarrow 0} \frac{[\nabla f(\mathbf{x} + \theta \mathbf{v}) \cdot \mathbf{v}] - \nabla f(\mathbf{x}) \cdot \mathbf{v}}{\|\mathbf{v}\|} \\
 &= \lim_{\|\mathbf{v}\| \rightarrow 0} [\nabla f(\mathbf{x} + \theta \mathbf{v}) - \nabla f(\mathbf{x})] \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} \\
 &= 0 & (\frac{\mathbf{v}}{\|\mathbf{v}\|} \text{ is a unit vector, remains 1})
 \end{aligned}$$

■

2.7 2nd Order Mean Value Theorem in \mathbb{R}^n

$f \in C^2$ on \mathbb{R}^n

$$f(\mathbf{x} + \mathbf{v}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{v} + \frac{1}{2} \mathbf{v}^T \nabla^2 f(\mathbf{x} + \theta \mathbf{v}) \cdot \mathbf{v}$$

Remarks In this course, ∇^2 means Hessian, not Laplacian.

$$\nabla^2 f(\mathbf{x}) = \left(\frac{\partial^2 f}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \right)_{1 \leq i, j \leq n}(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial_1^2} & \frac{\partial^2 f}{\partial_1 \partial_2} & \cdots \\ \frac{\partial^2 f}{\partial_2 \partial_1} & \cdots & \\ \vdots & & \end{pmatrix}$$

The Hessian matrix is [symmetric](#). This is sometimes called Clairaut's Theorem.

note: $\mathbf{v}^T \nabla^2 f(\mathbf{x}) \mathbf{v} = \sum_{1 \leq i, j \leq n} \frac{\partial^2 f}{\partial \mathbf{x}_i \partial \mathbf{x}_j} f(\mathbf{x}) \mathbf{v}_i \mathbf{v}_j$

2.8 2nd Order Taylor Approximation in \mathbb{R}^n

$f \in C^2$ on \mathbb{R}^n

$$f(\mathbf{x} + \mathbf{v}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{v} + \frac{1}{2} \mathbf{v}^T \nabla^2 f(\mathbf{x}) \mathbf{v} + o(\|\mathbf{v}\|^2)$$

proof:

$$\begin{aligned}
 \lim_{\|\mathbf{v}\| \rightarrow 0} \frac{[f(\mathbf{x} + \mathbf{v}) - f(\mathbf{x})] - \nabla f(\mathbf{x}) \cdot \mathbf{v} - \frac{1}{2} \mathbf{v}^T \nabla^2 f(\mathbf{x}) \mathbf{v}}{\|\mathbf{v}\|^2} &= \lim_{\|\mathbf{v}\| \rightarrow 0} \frac{[\frac{1}{2} \mathbf{v}^T \nabla^2 f(\mathbf{x} + \theta \mathbf{v}) \cdot \mathbf{v}] - \frac{1}{2} \mathbf{v}^T \nabla^2 f(\mathbf{x}) \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \\
 &\quad \text{(By 2nd MVT)} \\
 &= \lim_{\|\mathbf{v}\| \rightarrow 0} \frac{1}{2} \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right)^T [\nabla^2 f(\mathbf{x} + \theta \mathbf{v}) - \nabla^2 f(\mathbf{x})] \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\
 &= 0
 \end{aligned}$$

■

2.9 Geometric Meaning of Gradient

$f : \mathbb{R}^n \rightarrow \mathbb{R}$

Rate of change of f at \mathbf{x} in direction \mathbf{v} ($\|\mathbf{v}\| = 1$) = $\frac{d}{dt} |_{t=0} f(\mathbf{x} + t\mathbf{v})$

$$\begin{aligned}
 \frac{d}{dt} |_{t=0} f(\mathbf{x} + t\mathbf{v}) &= \nabla f(\mathbf{x} + t\mathbf{v}) \cdot \mathbf{v} |_{t=0} \\
 &= \nabla f(\mathbf{x}) \cdot \mathbf{v} \\
 &= |\nabla f(\mathbf{x})| |\mathbf{v}| \cos \theta \\
 &= |\nabla f(\mathbf{x})| \cos \theta \quad (\|\mathbf{v}\| = 1)
 \end{aligned}$$

maximized at $\theta = 0$

So $\nabla f(\mathbf{x})$ points in the direction of steepest ascent.

2.10 Implicit Function Theorem

$f : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \in C^1$

Fix $(\mathbf{a}, b) \in \mathbb{R}^n \times \mathbb{R}$ s.t. $f(\mathbf{a}, b) = 0$.

If $\nabla f(\mathbf{a}, b) \neq 0$, then $\{(\mathbf{x}, y) \in (\mathbb{R}^n \times \mathbb{R}) | f(\mathbf{x}, y) = 0\}$ is locally (near (\mathbf{a}, b)) the graph of a function.

2.11 Level Sets of f

c -level set of $f := \{\mathbf{x} \in \mathbb{R}^n | f(\mathbf{x}) = c\}$

Fact gradient $\nabla f(\mathbf{x}_0) \perp$ level curve (through \mathbf{x}_0)

3 Convex Sets & Functions

3.1 Definitions

Definition of Convex Set $\Omega \subseteq \mathbb{R}^n$ is a convex set if $\mathbf{x}_1, \mathbf{x}_2 \in \Omega \Rightarrow s\mathbf{x}_1 + (1-s)\mathbf{x}_2 \in \Omega$ where $s \in [0, 1]$

Definition of Convex Function A function $f : \text{convex } \Omega \subseteq \mathbb{R}^n$ is convex if

$$f(s\mathbf{x}_1 + (1-s)\mathbf{x}_2) \leq sf(\mathbf{x}_1) + (1-s)f(\mathbf{x}_2)$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in \Omega$ and all $s \in [0, 1]$

Remarks Second line above (or equal to) the graph

Definition of Concave Function A function f is concave if $-f$ is convex.

3.2 Basic Properties of Convex Functions

Let $\Omega \subseteq \mathbb{R}^n$ be a convex set.

1. f_1, f_2 are convex functions on $\Omega \Rightarrow f_1 + f_2$ is a convex function on Ω .
2. f is a convex function, $a \geq 0 \Rightarrow af$ is a convex function.
3. f is a convex function on $\Omega \Rightarrow$ The sublevel sets of f , $SL_c := \{\mathbf{x} \in \mathbb{R}^n | f(\mathbf{x}) \leq c\}$ is convex.

proof of (3):

Let $x_1, x_2 \in SL_c$, so that $f(x_1) \leq c$ and $f(x_2) \leq c$.

WTS: $sx_1 + (1-s)x_2 \in SL_c$ for any $s \in [0, 1]$

$$\begin{aligned}
 f(sx_1 + (1-s)x_2) &\leq sf(x_1) + (1-s)f(x_2) && (f \text{ is convex}) \\
 &\leq sc + (1-s)c \\
 &= c \\
 \Rightarrow sx_1 + (1-s)x_2 &\in SL_c
 \end{aligned}$$

■

Example of a convex function Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = |x|$

Let $x_1, x_2 \in \mathbb{R}$, $s \in [0, 1]$

Then

$$\begin{aligned}
 f(sx_1 + (1-s)x_2) &= |sx_1 + (1-s)x_2| \\
 &\leq |sx_1| + |(1-s)x_2| && (\text{by Triangle Inequality}) \\
 &= s|x_1| + (1-s)|x_2| \\
 &= sf(x_1) + (1-s)f(x_2)
 \end{aligned}$$

Then f is a convex function.

Theorem - Characterization of C^1 convex functions Let $f : \text{convex subset of } \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 function.

Then,

$$f \text{ is convex} \iff f(y) \geq f(x) + \nabla f(x) \cdot (y - x) \text{ for all } x, y \in \Omega$$

Remarks Tangent line below the graph.

proof:

(\Rightarrow)

f is convex, then by definition,

$$\begin{aligned}
 f(s\mathbf{x}_1 + (1-s)\mathbf{x}_2) &\leq sf(\mathbf{x}_1) + (1-s)f(\mathbf{x}_2) \\
 f(s\mathbf{x}_1 + (1-s)\mathbf{x}_2) - f(\mathbf{x}_2) &\leq s(f(\mathbf{x}_1) - f(\mathbf{x}_2)) \\
 \frac{f(s\mathbf{x}_1 + (1-s)\mathbf{x}_2) - f(\mathbf{x}_2)}{s} &\leq f(\mathbf{x}_1) - f(\mathbf{x}_2) \\
 \lim_{s \rightarrow 0} \frac{f(\mathbf{x}_2 + s(\mathbf{x}_1 - \mathbf{x}_2)) - f(\mathbf{x}_2)}{s} &\leq f(\mathbf{x}_1) - f(\mathbf{x}_2) \\
 \nabla f(\mathbf{x}_2) \cdot (\mathbf{x}_1 - \mathbf{x}_2) &\leq f(\mathbf{x}_1) - f(\mathbf{x}_2) \quad (\text{since } \frac{d}{ds}|_{s=0} f(\mathbf{x}_2 + s(\mathbf{x}_1 - \mathbf{x}_2)) = \nabla f(\mathbf{x}_2) \cdot (\mathbf{x}_1 - \mathbf{x}_2)) \\
 f(\mathbf{x}_2) + \nabla f(\mathbf{x}_2) \cdot (\mathbf{x}_1 - \mathbf{x}_2) &\leq f(\mathbf{x}_1) \\
 f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) &\leq f(\mathbf{y})
 \end{aligned}$$

where $0 \leq s \leq 1$

(\Leftarrow)

Fix $\mathbf{x}_0, \mathbf{x}_1 \in \Omega$ and $s \in (0, 1)$

Let $x = s\mathbf{x}_0 + (1 - s)\mathbf{x}_1$

$$\begin{cases} f(\mathbf{x}_0) & \geq f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot (\mathbf{x}_0 - \mathbf{x}) \\ & = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot (1 - s)(\mathbf{x}_0 - \mathbf{x}_1) \\ f(\mathbf{x}_1) & \geq f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot (\mathbf{x}_1 - \mathbf{x}) \\ & = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot s(\mathbf{x}_1 - \mathbf{x}_0) \end{cases}$$

$$\begin{cases} sf(x_0) & \geq sf(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot (1 - s) \cdot s(\mathbf{x}_0 - \mathbf{x}_1) \\ (1 - s)f(\mathbf{x}_1) & \geq (1 - s)f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot (1 - s) \cdot s(\mathbf{x}_1 - \mathbf{x}_0) \end{cases}$$

Then

$$sf(\mathbf{x}_0) + (1 - s)f(\mathbf{x}_1) \geq f(x) + 0$$

Then f is convex. ■

3.3 Criteria for convexity

C^1 criterion for convexity

$$f : \Omega \rightarrow \mathbb{R} \text{ is convex} \iff f(y) \geq f(x) + \nabla f(x) \cdot (y - x)$$

for all $x, y \in \Omega$

Theorem: C^2 criterion for convexity Let $f \in C^2$ on $\Omega \subseteq \mathbb{R}^n$ (here we assume $\Omega \subseteq \mathbb{R}^n$ is a convex set containing an interior point)

Then

$$f \text{ is convex on } \Omega \iff \nabla^2 f(x) \geq 0$$

for all $x \in \Omega$

Remark 1 Let A be an $n \times n$ matrix.

“ $A \geq 0$ ” means A is positive semi-definite:

$$v^T A v \geq 0$$

for all $v \in \mathbb{R}^n$

Remark 2 In \mathbb{R} ,

$$f \text{ is convex} \iff f'(x) \geq 0$$

for all $x \in \Omega$

(“concave up” in first year calculus)

proof for Theorem:

Recall 2nd order MVT:

$$f(y) = f(x) + \nabla f(x) \cdot (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(x + s(y - x)) \cdot (y - x)$$

for some $s \in [0, 1]$

(\Leftarrow)

Since $\nabla^2 f(x) \geq 0$, then

$$\frac{1}{2}(y - x)^T \nabla^2 f(x + s(y - x)) \cdot (y - x) \geq 0$$

Then

$$f(y) \geq f(x) + \nabla f(x) \cdot (y - x)$$

for all $x, y \in \Omega$.

Then by C^1 criterion, f is convex.

(\Rightarrow)

Assume f is convex on Ω .

Suppose for contradiction that $\nabla^2 f(x)$ is not positive semi-definite at some $x \in \Omega$.

Then $\exists v \neq 0$ s.t. $v^T \nabla^2 f(x) v < 0$ v could be arbitrarily small and > 0

Let $y = x + v$, then

$$(y - x)^T \nabla^2 f(x + s(y - x)) \cdot (y - x) < 0$$

for all $s \in [0, 1]$

Then by MVT,

$$f(y) < f(x) + \nabla f(x) \cdot (y - x)$$

for some $x, y \in \Omega$, and this contradicts the C^1 criterion. ■

3.4 Minimization and Maximization of Convex Functions

Theorem $f : \text{convex } \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function.

Suppose $\Gamma := \{x \in \Omega | f(x) = \min_{\Omega} f(x)\} \neq \emptyset$

(i.e. minimizer exists)

Then Γ is a convex set, and any local minimum of f is a global minimum of f .

proof:

Let $m = \min_{\Omega} f(x)$.

$$\Gamma = \{x \in \Omega | f(x) = m\} = \{x \in \Omega | f(x) \leq m\}$$

(sublevel set)

Then by Basic Properties of Convex Sets, Γ is convex.

Let x be a local minimum of f .

Suppose for contradiction that $\exists y$ s.t. $f(y) < f(x)$

(i.e. x is not a global minimum)

$$\begin{aligned} f(sy + (1-s)x) &\leq sf(y) + (1-s)f(x) \\ &< sf(x) + (1-s)f(x) && (f(y) < f(x)) \\ &= f(x) \end{aligned}$$

for all $s \in (0, 1)$

As s approaches 0, s approaches x .

Then we have $\lim_{s \rightarrow 0} f(sy + (1-s)x) = f(x) < f(x)$.

which is a contradiction. ■

Theorem If $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function, and Ω is convex and compact, then

$$\max_{\Omega} f = \max_{\partial\Omega} f$$

Remarks Maximum value of f is attained (also) on the boundary of Ω

proof:

Since Ω is closed, $\partial\Omega \subseteq \Omega$, so $\max_{\Omega} f \geq \max_{\partial\Omega} f$.

Suppose $f(x_0) = \max_{\Omega} f$ for some $x_0 \notin \partial\Omega$. Let L be an arbitrary line through x_0 .

By convexity and compactness of Ω , L meets $\partial\Omega$ at two points x_1, x_2 .

Let $x_0 + sx_1 + (1-s)x_2$ for $s \in (0, 1)$

$$\begin{aligned}
 f(x_0) &= f(sx_1 + (1-s)x_2) \\
 &\leq sf(x_1) + (1-s)f(x_2) \\
 &\leq \max\{f(x_1), f(x_2)\} \\
 &\leq \max_{\partial\Omega} f \\
 &\leq \max_{\Omega} f = f(x_0)
 \end{aligned}
 \tag{f convex}$$

This implies that

$$\max_{\Omega} f = \max_{\partial\Omega} f$$

as wanted. ■

Example

$$|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$$

where $p, q > 1$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

Special cases:

1.

$$p = q = 2, |ab| \leq \frac{|a|^2 + |b|^2}{2}$$

2.

$$p = 3, q = \frac{3}{2}, |ab| \leq \frac{1}{3}|a|^3 + \frac{2}{3}|b|^{\frac{3}{2}}$$

proof:

Since function $f(x) = -\log(x)$ is convex, then

$$\begin{aligned}
 (-\log)|ab| &= (-\log)|a| + (-\log)|b| \\
 &= \frac{1}{p}(-\log)|a|^p + \frac{1}{q}(-\log)|b|^q \\
 &\geq (-\log)\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right) \\
 (-\log)|ab| &\geq (-\log)\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right) \\
 \log|ab| &\leq \log\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right) \\
 |ab| &\leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q
 \end{aligned}
 \tag{exponential function is increasing}$$

■

4 Basics of Unconstrained Optimization

4.1 Extreme Value Theorem

Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous, and compact set $K \subseteq \mathbb{R}^n$. Then the problem

$$\min_{x \in K} f(x)$$

has a solution.

Recall

1.

$$K \subseteq \mathbb{R}^n \text{ compact} \iff K \text{ closed and bounded}$$

2. If h_1, \dots, h_k and g_1, \dots, g_m are continuous functions on \mathbb{R}^n , then the set of all points $x \in \mathbb{R}^n$ s.t.

$$\begin{cases} h_i(x) = 0 & \text{for all } i \\ g_j(x) \leq 0 & \text{for all } j \end{cases}$$

is a closed set.

3. If such a set is also bounded, then it is compact.

Example

$$\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 - 1 = 0\}$$

by (2), this is a closed set

by (3), this is a compact set.

Remarks $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ convex does not imply f is continuous.**4.2 Unconstrained Optimization**

$$\min_{x \in \Omega \subseteq \mathbb{R}^n} f(x)$$

typically

1. $\Omega \subseteq \mathbb{R}^n$ 2. $\Omega = \mathbb{R}^n$ 3. $\Omega = \text{open}$ 4. $\Omega = \overline{\text{open}}$ **Remark**1. $\max f(x) = -(\min -f(x))$ 2. $\min f(x) = -(\max -f(x))$ **Definition: local minimum** We say that f has a local minimum at a point $x_0 \in \Omega$ if

$$f(x_0) \leq f(x)$$

for all $x \in B_\Omega^\varepsilon(x_0)$, where $B_\Omega^\varepsilon(x_0) = \{x \in \Omega : |x - x_0| < \varepsilon\}$ which is an open ball around x_0 inside Ω of radius $\varepsilon > 0$.We say that f has a strict local minimum at a point $x_0 \in \Omega$ if

$$f(x_0) < f(x)$$

for all $x \in B_\Omega^\varepsilon(x_0) \setminus \{x_0\}$

4.3 1st order necessary condition for local minimum

Theorem Let f be a C^1 function on $\Omega \subseteq \mathbb{R}^n$. If $x_0 \in \Omega$ is a local minimum of f , then

$$\nabla f(x_0) \cdot v \geq 0$$

for all feasible directions v at x_0

Definition: feasible direction $v \in \mathbb{R}^n$ is a feasible direction at $x_0 \in \Omega$ if

$$x_0 + sv \in \Omega$$

for all $0 \leq s \leq \bar{s}$ where $\bar{s} \in \mathbb{R}$

Remarks Feasible directions go into the set.

Corollary Special case: If $\Omega = \mathbb{R}^n$ is an open set, then any direction is a feasible direction. Then x_0 is a local minimum of f on Ω implies that $\nabla f(x_0) \cdot v \geq 0$ for all $v \in \mathbb{R}^n$.

$$\begin{aligned} \begin{cases} \nabla f(x_0) \cdot v \geq 0 \\ \nabla f(x_0) \cdot (-v) \geq 0 \end{cases} &\iff \nabla f(x_0) \cdot v \leq 0 \implies \nabla f(x_0) \cdot v = 0 \text{ for all } v \in \mathbb{R}^n \\ &\implies \nabla f(x_0) = 0 \end{aligned}$$

4.4 2nd order necessary condition for local minimum

$f \in C^2, \Omega \subseteq \mathbb{R}^n$

If $x_0 \in \Omega$ is a local minimum of f on Ω , then

1. $\nabla f(x_0) \cdot v \geq 0$ for all feasible directions v at x_0
2. If $\nabla f(x_0) \cdot v = 0$, then $v^T \nabla^2 f(x_0) v \geq 0$ (function curves up)

Remark If x_0 is an interior point of Ω , then

$$\begin{aligned} \nabla f(x_0) &= 0, \quad \nabla^2 f(x_0) \geq 0 \\ f'(x_0) &= 0, \quad f''(x_0) \geq 0 \end{aligned}$$

Definition: principal minor Let A be an $n \times n$ matrix. A $k \times k$ submatrix of A formed by deleting $n - k$ rows of A , and the **same** $n - k$ columns of A , is called principal submatrix of A . The determinant of a principal submatrix of A is called a principal minor of A .

Definition: leading principal minor Let A be an $n \times n$ matrix. The k th order principal submatrix of A obtained by deleting the **last** $n - k$ rows and columns of A is called the k -th order leading principal submatrix of A , and its determinant is called a leading principal minor of A .

Definition: positive definiteness (Sylvester's Criterion) A $n \times n$ matrix A is

1. positive definite if $v^T A v > 0$ for all $v \neq 0 \iff$ all eigenvalues $> 0 \iff$ **all leading principle minors > 0**
2. positive semi-definite if $v^T A v \geq 0$ for all $v \iff$ all eigenvalues $\geq 0 \iff$ **all principle minors ≥ 0**

Lemma Suppose $\nabla^2 f(x_0)$ is positive definite, then

$$\exists a > 0 \text{ s.t. } v^T \nabla^2 f(x_0) v \geq a \|v\|^2 \quad \forall v$$

4.5 2nd order sufficient condition (for interior points)

$f \in C^2$ on Ω

If $\begin{cases} \nabla f(x_0) = 0 \\ \nabla^2 f(x_0) > 0 \end{cases}$, then x_0 is a strict local minimum.

■

5 Optimization with Equality Constraints

5.1 Definitions of Related Spaces

Definition 5.1.1: surface

$$M = \text{“surface”} = \{x \in \mathbb{R}^n | h_1(x) = 0, \dots, h_k(x) = 0\}$$

where $h_i \in C^1$

Definition 5.1.2: differentiable curve on surface A differentiable curve on surface $M \subseteq \mathbb{R}^n$ is a C^1 function

$$x : (-\epsilon, \epsilon) \rightarrow M : s \mapsto x(s)$$

Remarks

1. Let $x(s)$ be a differentiable curve on M that passes through $x_0 \in M$, say $x(0) = x_0$. The vector $v = \frac{d}{ds}|_{s=0} x(0)$ touches M “tangentially”. We say v is generated by $x(s)$.
2. In previous calculus courses, differentiable curves are often referred to as parameterizations.

Definition 5.1.3: tangent vector Any vector v which is generated by some differentiable curve on M through x_0 is called a tangent vector.

Definition 5.1.4: tangent space Tangent space to the surface M at point x_0 is

$$T_{x_0} M = \{\text{all tangent vectors to } M \text{ at } x_0\} = \{v \in \mathbb{R}^n : v = \frac{d}{ds}|_{s=0} x(s)\}$$

where $x(s)$ is a differentiable curve on M s.t. $x(0) = x_0$

Remarks The zero vector is contained in all tangent spaces.

Definition 5.1.5: T-space

$$T_{x_0} = \{x \in \mathbb{R}^n : x^T \nabla h_i(x_0) = 0 \forall i\} = \text{Span}\{\nabla h_1(x_0), \dots, \nabla h_k(x_0)\}^\perp$$

Definition 5.1.6: regular point $x_0 \in M$ is a regular point (of the constraints) if $\{\nabla h_1(x_0), \dots, \nabla h_k(x_0)\}$ are linearly independent.

Remark If there is only one constraint h , then x_0 is regular if and only if $\nabla h(x_0) \neq 0$.

When does the T-space equivalent to the tangent space? When x_0 is a regular point (of the constraints).

Theorem 5.1.7 Suppose x_0 is a regular point s.t. $M = \{x \in \mathbb{R}^n : h_i(x) = 0 \forall i\}$. Then

$$T_{x_0}M = T_{x_0}$$

Lemma 5.1.8 $f, h_1, \dots, h_k \in C^1$ on open $\Omega \subseteq \mathbb{R}^n$

$$M = \{x \in \mathbb{R}^n : h_i(x) = 0 \forall i\}$$

Suppose $x_0 \in M$ is a local minimum of f on M , then

$$\nabla f(x_0) \perp T_{x_0}M \iff \nabla f(x_0) \cdot v = 0$$

for all $v \in T_{x_0}M$

5.2 Lagrange Multipliers: 1st order necessary condition for local minimum

$f, h_1, \dots, h_k \in C^1$ on open $\Omega \subseteq \mathbb{R}^n$.

Let x_0 be a regular point of the constraints $M = \{x \in \mathbb{R}^n : h_i(x) = 0 \forall i\}$.

Suppose x_0 is a local minimum of f on M , then $\exists \lambda_1, \dots, \lambda_k \in \mathbb{R}$ s.t.

$$\nabla f(x_0) + \lambda_1 \nabla h_1(x_0) + \dots + \lambda_k \nabla h_k(x_0) = 0$$

Proof. x_0 regular implies that

$$T_{x_0}M = T_{x_0} = \text{Span}\{\nabla h_1(x_0), \dots, \nabla h_k(x_0)\}^\perp$$

By Lemma 5.1.8, x_0 is a loc min implies that

$$\nabla f(x_0) \perp T_{x_0}M$$

Then

$$\nabla f(x_0) \in (T_{x_0}M)^\perp = \text{Span}\{\nabla h_i(x_0)\}^{\perp\perp} = \text{Span}\{\nabla h_i(x_0)\}$$

Then

$$\nabla f(x_0) = -\lambda_1 \nabla h_1(x_0) - \dots - \lambda_k \nabla h_k(x_0)$$

for some $\lambda_i \in \mathbb{R}$ ■

5.3 2nd order necessary condition for local minimum

$f, h_1, \dots, h_k \in C^2$ on open $\Omega \subseteq \mathbb{R}^n$.

Let x_0 be a regular point of the constraints $M = \{x \in \mathbb{R}^n : h_i(x) = 0 \forall i\}$.

Suppose x_0 is a local minimum of f on M , then

1.

$$\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) = 0$$

for some $\lambda_i \in \mathbb{R}$

2.

$$\nabla^2 f(x_0) + \sum \lambda_i \nabla^2 h_i(x_0) \geq 0$$

on $T_{x_0}M$

5.4 2nd order sufficient condition for local minimum

$f, h_1, \dots, h_k \in \mathcal{C}^2$ on open $\Omega \subseteq \mathbb{R}^n$.

Let x_0 be a regular point of the constraints $M = \{x \in \mathbb{R}^n : h_i(x) = 0 \forall i\}$.

If $\exists \lambda_i \in \mathbb{R}$ s.t.

1.

$$\nabla f(x_0) + \sum \lambda_i \nabla h_i(x_0) = 0$$

2.

$$\nabla^2 f(x_0) + \sum \lambda_i \nabla^2 h_i(x_0) > 0$$

on $T_{x_0}M$

Then x_0 is a strict local minimum.

Proof. Recall that (2) means $[\nabla^2 f(x_0) + \sum \lambda_i \nabla^2 h_i(x_0)]$ is pos-def on $T_{x_0}M$.

Then $\exists a > 0$ s.t. $v^T [\nabla^2 f(x_0) + \sum \lambda_i \nabla^2 h_i(x_0)] v \geq a \|v\|^2$ for all $v \in T_{x_0}M$.

Let $x(s) \in M$ be a curve s.t. $x(0) = x_0$ and $v = x'(0)$.

WLOG, $\|x'(0)\| = 1$.

By 2nd order Taylor,

$$\begin{aligned} f(x(s)) - f(x(0)) &= s \frac{d}{ds} \Big|_{s=0} f(x(s)) + \frac{1}{2} s^2 \frac{d^2}{ds^2} \Big|_{s=0} f(x(s)) + o(s^2) \\ &= s \frac{d}{ds} \Big|_{s=0} [f(x(s)) + \sum_i \lambda_i h_i(x(s))] + \frac{1}{2} s^2 \frac{d^2}{ds^2} \Big|_{s=0} [f(x(s)) + \sum_i \lambda_i h_i(x(s))] + o(s^2) \\ &\quad (\sum_i \lambda_i h_i(x(s)) = 0) \\ &= s [\nabla f(x_0) + \sum \lambda_i \nabla h_i(x_0)] \cdot x'(0) + \frac{1}{2} s^2 x'(0)^T [\nabla^2 f(x_0) + \sum \lambda_i \nabla^2 h_i(x_0)] x'(0) + o(s^2) \\ &= 0 + \frac{1}{2} s^2 v^T [\nabla^2 f(x_0) + \sum \lambda_i \nabla^2 h_i(x_0)] v + o(s^2) \\ &\geq \frac{1}{2} s^2 a \|v\|^2 + o(s^2) \\ &= \frac{1}{2} s^2 a + o(s^2) \\ &= s^2 \left[\frac{a}{2} + \frac{o(s^2)}{s^2} \right] > 0 \end{aligned}$$

for small $s > 0$, since $\frac{a}{2} > 0$ and $\lim_{s \rightarrow 0} \frac{o(s^2)}{s^2} = 0$

Then $f(x(s)) > f(x_0)$ for small $s > 0$ Then x_0 is a strict local min of f . ■

6 Optimization with Inequality Constraints

Problem open $\Omega \subseteq \mathbb{R}^n$

$f : \Omega \rightarrow \mathbb{R}$

$h_1, \dots, h_k : \Omega \rightarrow \mathbb{R}$

$g_1, \dots, g_l : \Omega \rightarrow \mathbb{R}$

$$\begin{cases} \min f(x) \\ x \in \Omega \text{ subject to } \begin{cases} h_1(x) = 0, \dots, h_k(x) = 0 \\ g_1(x) \leq 0, \dots, g_l(x) \leq 0 \end{cases} \end{cases} \quad (*)$$

Definition 1: activeness Let x_0 satisfy the constraints.
 We say that the constraint $g_i(x) \leq 0$ is active at x_0 if $g_i(x_0) = 0$.
 It is inactive at x_0 if $g_i(x_0) < 0$.

Definition 2: regular point Suppose for some $l' \leq l$:

$$g_1(x) \leq 0, \dots, g_{l'}(x) \leq 0; g_{l'+1}(x) \leq 0, \dots, g_l(x) \leq 0$$

where $g_1, \dots, g_{l'}$ active and the rest inactive.

We say that x_0 is a regular point of the constraints if
 $\{\nabla h_1(x_0), \dots, \nabla h_k(x_0), \nabla g_1(x_0), \dots, \nabla g_{l'}(x_0)\}$ is linearly independent.

6.1 Kuhn-Tucker conditions: 1st order necessary condition for local minimum

open $\Omega \subseteq \mathbb{R}^n$

$f : \Omega \rightarrow \mathbb{R}$

$h_1, \dots, h_k, g_1, \dots, g_l : C^1 \in \Omega$

Suppose $x_0 \in \Omega$ is a regular point of the constraints which is a local minimum, then

1.

$$\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j=1}^l \mu_j \nabla g_j(x_0) = 0$$

for some $\lambda_i \in \mathbb{R}$ and $\mu_j \geq 0$

2. $\mu_j g_j(x_0) = 0$

Remark 1 Given x_0 ,

$$\begin{cases} g_j(x) \leq 0 \text{ active at } x_0 \implies g_j(x_0) = 0 \implies \mu_j g_j(x_0) = 0 \\ g_j(x) \leq 0 \text{ inactive at } x_0 \implies g_j(x_0) < 0 \implies \mu_j = 0 \end{cases}$$

$\implies \mu_j = 0$ for all inactive g_j at x_0

Remark 2 It is possible for an active constraint to have zero multiplier.

Remark 3 $\mu_j \geq 0$ because ∇f and ∇g have opposite directions at a local minimum x_0 .

$$\nabla f(x_0) + \mu \nabla g(x_0) = 0 \implies \nabla f(x_0) = -\mu \nabla g(x_0) \implies -\mu < 0 \implies \mu > 0$$

Is this true?

Idea of proof x_0 is a local min of f subject to (*)

$\implies x_0$ is a local min for equality constraints $h_1(x) = 0, \dots, h_k(x) = 0$ + active inequality constraints $g_1(x) \leq 0, \dots, g_{l'}(x) \leq 0$

$\implies x_0$ is a local min for $h_1(x) = 0, \dots, h_k(x) = 0 + g_1(x) = 0, \dots, g_{l'}(x) = 0 \implies \nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j=1}^{l'} \mu_j \nabla g_j(x_0) = 0$

for some $\lambda_i \in \mathbb{R}$ and $\mu_j \in \mathbb{R}$.

Let $\mu_j = 0$ for $j = l' + 1, \dots, l$, then

$$\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j=1}^l \mu_j \nabla g_j(x_0) = 0$$

6.2 2nd order necessary conditions for local minimum

Open $\Omega \subseteq \mathbb{R}^n$, $f, h_1, \dots, h_k, g_1, \dots, g_l \in C^2$. Let x_0 be a regular point of the constraints:

$$(\dagger) \begin{cases} h_1(x) = \dots = h_k(x_0) = 0 \\ g_1(x), \dots, g_l(x_0) \leq 0 \end{cases}$$

Suppose x_0 is a local min of f subject to (\dagger) . Then, $\exists \lambda_i \in \mathbb{R}, \mu_j \geq 0$ s.t.

1. $\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j=1}^l \mu_j \nabla g_j(x_0) = 0$
2. $\mu_j g_j(x_0) = 0$ for all j
3. $[\nabla^2 f(x_0) + \sum \lambda_i \nabla^2 h_i(x_0) + \sum \mu_j \nabla^2 g_j(x_0)]$ is positive semi definite on tangent space to active constraints at x_0 .

Proof. x_0 local min for (\dagger)

$\implies x_0$ local min for only active constraints at x_0 .

$$\implies \begin{cases} h_i(x) = 0 \quad \forall i \\ g_j(x) = 0 \quad j = 1, \dots, l' \end{cases}$$

$\implies [\nabla^2 f(x_0) + \sum \lambda_i \nabla^2 h_i(x_0) + \sum \mu_j \nabla^2 g_j(x_0)]$ pos semi def on tangent space to active constraints. ■

6.3 2nd order sufficient conditions

Open $\Omega \subseteq \mathbb{R}^n$, $f, h_i, g_j \in C^2$ on Ω .

Problem:

$$\begin{cases} \min & f(x) \\ \text{subject to} & \begin{cases} h_i(x) = 0 \\ g_j(x) \leq 0 \end{cases} \end{cases}$$

Suppose $\exists x_0$ feasible and $\lambda_i, \mu_j \in \mathbb{R}$ s.t.

1. $\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j=1}^l \mu_j \nabla g_j(x_0) = 0$
2. $\mu_j g_j(x_0) = 0$ all j

If the Hessian matrix, $L(x_0) = \nabla^2 f(x_0) + \sum_{i=1}^k \lambda_i \nabla^2 h_i(x_0) + \sum_{j=1}^l \mu_j \nabla^2 g_j(x_0)$ is pos def on \tilde{T}_{x_0} -space of “strongly active” constraints at x_0 .

Then x_0 is a strict local min.

Remarks

1.

$$\text{Active constraints at } x_0 \begin{cases} h_i(x) = 0 & i = 1, \dots, k \\ g_j(x) \leq 0 & j = 1, \dots, l' \end{cases} \implies g_j(x_0) = 0$$

2.

$$\text{Strongly active constraints at } x_0 \begin{cases} h_i(x) = 0 & i = 1, \dots, k \\ g_j(x) \leq 0 & j = 1, \dots, l'' \end{cases} \quad g_j(x) \text{ is active at } x_0 \text{ and } \mu_j > 0$$

$$l'' \leq l' \leq l$$

3.

$$\tilde{T}_{x_0} = \{v \in \mathbb{R}^n \mid v \cdot \nabla h_i(x_0) = 0 \text{ all } i \text{ and } v \cdot \nabla g_j(x_0) = 0 \text{ for all } j = 1, \dots, l''\}$$

4. strongly active \subseteq active
 $\implies \tilde{T}_{x_0} = (\text{strongly active})^\perp \supseteq (\text{active})^\perp = \tilde{T}_{x_0}$

Proof. (details see another pdf by prof) Suppose x_0 is **NOT** a (strict) local min.

claim: \exists unit vector $v \in \mathbb{R}$ s.t.

1. $\nabla f(x_0) \cdot v \leq 0$
2. $\nabla h_i(x_0) \cdot v = 0 \quad i = 1, \dots, k$
3. $\nabla g_j(x_0) \cdot v \leq 0 \quad j = 1, \dots, l'$

proof of claim: \square

claim: $\nabla g_j(x) \cdot v = 0$ for $j = 1, \dots, l''$

proof of claim: \square

\implies contradiction!

claim: \exists unit vector $v \in \mathbb{R}$ s.t.

1. $\nabla f(x_0) \cdot v \leq 0$
2. $\nabla h_i(x_0) \cdot v = 0 \quad i = 1, \dots, k$
3. $\nabla g_j(x_0) \cdot v = 0 \quad j = 1, \dots, l''$

proof of claim: \square

■

7 Different Computation Methods for Solving Optimum

7.1 Newton's Method

$x_0 \in I$ start

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$

Theorem Let $f \in C^3$ on I .

Suppose $x_* \in I$ satisfies $f'(x_*) = 0$ and $f''(x_*) \neq 0$ (x_* is a non-degenerate (non-singular) critical point).

Then the sequence of points $\{x_n\}$ generated by Newton's method

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$

converges to x_* if x_0 is sufficiently close to x_* .

Why do we need this method? In real life, we may not know the real function formula. We only have data, using which we can approximate the function formula. In a way, Newton's method is true "applied mathematics".

Proof of Theorem Let $g(x) = f'(x)$ so that $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$

By $g \in C^2, \exists \alpha$ s.t. $|g'(x_1)| > \alpha \forall x_1$ and $|g''(x_2)| < \frac{1}{\alpha} \forall x_2$ in a neighbourhood of x_* (choose α small enough).

$$x_{n+1} - x_* = x_n - \frac{g(x_n)}{g'(x_n)} - x_* \quad (1)$$

$$= x_n - x_* - \frac{g(x_n) - g(x_*)}{g'(x_n)} \quad (g(x_*) = 0) \quad (2)$$

$$= \frac{-[g(x_n) - g(x_*) - g'(x_n)(x_n - x_*)]}{g'(x_n)} \quad (3)$$

$$= \frac{1}{2} \frac{g''(\xi)}{g'(x_n)} (x_n - x_*)^2 \quad (4)$$

$$|x_{n+1} - x_*| = \frac{1}{2} \frac{g''(\xi)}{g'(x_n)} |x_n - x_*|^2 < \frac{1}{2\alpha^2} |x_n - x_*|^2 \quad (\text{in small neighbourhood of } x_*) \quad (5)$$

$$\rho := \frac{1}{2\alpha^2} |x_0 - x_*| \quad (\text{choose } x_0 \text{ sufficiently close to } x_* \text{ s.t. } \rho < 1) \quad (6)$$

$$|x_1 - x_*| < \frac{1}{2\alpha^2} |x_0 - x_*|^2 \quad (7)$$

$$= \frac{1}{2\alpha^2} |x_0 - x_*| |x_0 - x_*| \quad (8)$$

$$= \rho |x_0 - x_*| \quad (9)$$

$$|x_2 - x_*| < \frac{1}{2\alpha^2} |x_1 - x_*|^2 \quad (10)$$

$$< \frac{1}{2\alpha^2} \rho^2 |x_0 - x_*|^2 \quad (11)$$

$$= \frac{1}{2\alpha^2} |x_0 - x_*| \rho^2 |x_0 - x_*| \quad (12)$$

$$< \rho^2 |x_0 - x_*| \quad (\rho < 1) \quad (13)$$

$$|x_n - x_*| < \rho^n |x_0 - x_*| \xrightarrow{n \rightarrow \infty} 0 \quad (14)$$

$$\implies x_n \rightarrow x_* \quad (15)$$

proof of (4):

By 2nd order MVT,

$$g(x) = g(y) + g'(y)(x - y) + \frac{1}{2} g''(\xi)(x - y)^2$$

for some $\xi \in [x, y]$.

Let $x = x_*$ and $y = x_n$, then

$$g(x_*) = g(x_n) + g'(x_n)(x_* - x_n) + \frac{1}{2} g''(\xi)(x_* - x_n)^2$$

$$\implies -[g(x_n) - g(x_*) - g'(x_n)(x_n - x_*)] = \frac{1}{2} g''(\xi)(x_n - x_*)^2$$

■

Newton's Method (generalized) $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ and $f \in C^3$ on Ω
open

$x_0 \in \Omega$

$x_{n+1} = x_n - [\nabla^2 f(x_n)]^{-1} \nabla f(x_n)$

(The algorithm requires $\nabla^2 f(x_n)$ invertible and stops when $\nabla f(x_n) = 0$)

Note Newton's method may fail to converge even if $f(x)$ has a unique global min x_* and x_0 is arbitrarily close to x_*

Remark Newton's method, if converge, converges to

1. local min
2. local max
3. saddle point

Example 7.1. Newton's Method on Quadratic Function Let Q be a symmetric $n \times n$ invertible matrix. Define quadratic form $f(x) := \frac{1}{2}x^T Qx : \mathbb{R}^n \rightarrow \mathbb{R}$. Then the optima is $x = 0$.

Let $x_0 \in \mathbb{R}^n$, then

$$x_1 := x_0 - \nabla^2 f(x_0)^{-1} \nabla f(x_0) = x_0 - Q^{-1} Q x_0 = 0$$

Newton's method converges in one iteration.

7.2 Method of Steepest Descent (Gradient Method)

$f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, C^1$
open

Recall: Direction of steepest ascent at x_0 is given by the direction of gradient $\nabla f(x_0)$

Algorithm of steepest descent $x_0 \in \Omega$

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

where $\alpha_k \geq 0$ satisfying $f(x_k - \alpha_k \nabla f(x_k)) = \min_{\alpha \geq 0} f(x_k - \alpha \nabla f(x_k))$

(keep going until you find the minimum)

Fact: algorithm is descending If $\nabla f(x_k) \neq 0$, then $f(x_{k+1}) < f(x_k)$

Why? $f(x_{k+1}) = f(x_k - \alpha_k \nabla f(x_k)) \leq f(x_k - \alpha \nabla f(x_k))$ for all $0 < \alpha \leq \alpha_k$

Recall: $\frac{d}{ds} \big|_{s=0} f(x_k - s \nabla f(x_k)) = \nabla f(x_k) \cdot (-\nabla f(x_k)) = -|\nabla f(x_k)|^2 < 0$

$\implies f(x_{k+1}) \leq f(x_k - \alpha \nabla f(x_k)) < f(x_k)$ for small α

Fact: the method of steepest descent moves perpendicular steps

$$(x_{k+2} - x_{k+1}) \cdot (x_{k+1} - x_k) = (-\alpha_{k+1} \nabla f(x_{k+1})) \cdot (-\alpha_k \nabla f(x_k)) \quad (16)$$

$$= \alpha_k \alpha_{k+1} \nabla f(x_{k+1}) \cdot \nabla f(x_k) \quad (17)$$

$$(18)$$

If $\alpha_k = 0$, then we are done.

If $\alpha_k \neq 0$, then

$$\nabla f(x_{k+1}) = \min_{\alpha \geq 0} \nabla f(x_k - \alpha \nabla f(x_k)) \quad (19)$$

$$\implies \frac{d}{d\alpha} \big|_{\alpha=\alpha_k} f(x_k - \alpha \nabla f(x_k)) = (-\nabla f(x_k)) \cdot \nabla f(x_k - \alpha_k \nabla f(x_k)) = 0 \quad (20)$$

$$\implies \alpha_k \alpha_{k+1} \nabla f(x_{k+1}) \cdot \nabla f(x_k) = 0 \quad (21)$$

Note This method is not the most efficient. May take infinite steps to converge.

Theorem (Convergence of Steepest Descent) $f \in C^1$ on $\Omega \underset{\text{open}}{\subseteq} \mathbb{R}^n$

Let $\{x_k\}$ be sequence generated by steepest descent.

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

If $\{x_k\}$ is “bounded in Ω ” (i.e. \exists compact set $K \subset \Omega$ s.t. $x_k \in K$ for all k)

Then every convergent subsequence of $\{x_k\}$ converges to a critical point $x_* \in \Omega$ of $f : \nabla f(x_*) = 0$

Proof. $x_k \in \text{compact } K \implies \text{subsequence } x_{k_i} \rightarrow x_* \in K$

Since $f(x_0) \geq f(x_1) \geq f(x_2) \geq \dots$ and $f(x_{k_i}) \searrow f(x_*)$

Suppose by contradiction that $\nabla f(x_*) \neq 0$

$$x_{k_i} \rightarrow x_* \implies \nabla f(x_{k_i}) \rightarrow \nabla f(x_*)$$

Let $y_{k_i} = x_{k_i} - \alpha_{k_i} \nabla f(x_{k_i}) = x_{k_i+1}$. Then $y_{k_i} \rightarrow y_*$. Then

$$f(y_{k_i}) = f(x_{k_i+1}) = \min_{\alpha \geq 0} f(x_i - \alpha \nabla f(x_{k_i})) \quad (22)$$

$$f(y_{k_i}) \leq f(x_{k_i} - \alpha \nabla f(x_{k_i})) \text{ for all } \alpha \geq 0 \quad (23)$$

$$\lim_{i \rightarrow \infty} f(y_{k_i}) \leq f(x_* - \alpha \nabla f(x_*)) \text{ for all } \alpha \geq 0 \quad (24)$$

$$f(y_*) \leq \min_{\alpha \geq 0} f(x_* - \alpha \nabla f(x_*)) < f(x_*) \quad (25)$$

$$f(y_*) < f(x_*) \quad (26)$$

$$(27)$$

But $f(y_*) = \lim_{i \rightarrow \infty} f(y_{k_i}) = \lim_{i \rightarrow \infty} f(x_{k_i+1}) = f(x_*)$, so we have a contradiction. ■

Steepest descent: Quadratic case Let f follow the general quadratic form

$$f(x) = \frac{1}{2} x^T Q x - b^T x$$

where $b, x \in \mathbb{R}^n$ and Q is an $n \times n$ positive definite matrix.

Let $0 < \lambda = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n = \Lambda$ be eigenvalues of Q .

Recall that if Q pos-def, then there is a unique minimum x_* such that $Qx_* - b = 0 \iff x_* = Q^{-1}b$

Define $q(x) := \frac{1}{2}(x - x_*)^T Q(x - x_*) = f(x) + \text{const}$

Note that $q(x) \geq 0$ and $q(x_*) = 0$.

Define $g(x) := Qx - b = \nabla q(x) = \nabla f(x)$

So using the method of steepest descent:

$$x_{k+1} = x_k - \alpha_k g(x_k)$$

Derive the formula for α_k :

α_k minimizes $f(x_k - \alpha g(x_k))$

$$0 = \frac{d}{d\alpha} \big|_{\alpha=\alpha_k} f(x_k - \alpha g(x_k)) \quad (28)$$

$$= \nabla f(x_k - \alpha_k g(x_k)) \cdot (-g(x_k)) \quad (29)$$

$$= -[Q(x_k - \alpha_k g(x_k)) - b] \cdot (g(x_k)) \quad (30)$$

$$= -(Qx_k - b) \cdot g(x_k) \quad (31)$$

$$= -|g(x_k)|^2 + \alpha_k g(x_k)^T Q g(x_k) \quad (32)$$

$$\implies \alpha_k = \frac{|g(x_k)|^2}{g(x_k)^T Q g(x_k)} \quad (33)$$

$$\implies x_{k+1} = x_k - \alpha_k g(x_k) \quad (34)$$

$$= x_k - \frac{|g(x_k)|^2}{g(x_k)^T Q g(x_k)} g(x_k) \quad (35)$$

Claim:

$$q(x_{k+1}) = \left(1 - \frac{|g(x_k)|^4}{(g(x_k)^T Q g(x_k))(g(x_k)^T Q^{-1} g(x_k))}\right) g(x_k)$$

Proof.

$$q(x_{k+1}) = q(x_k - \alpha_k g(x_k)) \quad (36)$$

$$= \frac{1}{2}(x_k - \alpha_k g(x_k) - x_*)^T Q (x_k - \alpha_k g(x_k) - x_*) \quad (37)$$

$$= \frac{1}{2}(x_k - x_* - \alpha_k g(x_k))^T Q ((x_k - x_*) - \alpha_k g(x_k)) \quad (38)$$

$$= \frac{1}{2}(x_k - x_*)^T Q (x_k - x_*) - \alpha_k g(x_k)^T Q (x_k - x_*) + \frac{1}{2}\alpha_k^2 g(x_k)^T Q g(x_k) \quad (39)$$

$$= q(x_k) - \alpha_k g(x_k)^T Q (x_k - x_*) + \frac{1}{2}\alpha_k^2 g(x_k)^T Q g(x_k) \quad (40)$$

$$\implies q(x_k) - q(x_{k+1}) = -\frac{1}{2}\alpha_k^2 g(x_k)^T Q g(x_k) + \alpha_k g(x_k)^T Q (x_k - x_*) \quad (41)$$

$$y_k := x_k - x_* \quad (42)$$

$$\frac{q(x_k) - q(x_{k+1})}{q(x_k)} = \frac{-\frac{1}{2}\alpha_k^2 g(x_k)^T Q g(x_k) + \alpha_k g(x_k)^T Q y_k}{\frac{1}{2}y_k^T Q y_k} \quad (43)$$

$$= \frac{2\alpha_k g(x_k)^T Q y_k - \alpha_k^2 g(x_k)^T Q g(x_k)}{y_k^T Q y_k} \quad (44)$$

$$(g_k := g(x_k) = Qx_k - b = Qx_k - Qx_* = Q(x_k - x_*) = Qy_k \implies y_k = Q^{-1}g_k) \quad (45)$$

$$= \frac{2\alpha_k |g_k|^2 - \alpha_k^2 g_k^T Q g_k}{g_k^T Q^{-1} g_k} \quad (46)$$

$$= \frac{2 \frac{|g_k|^4}{g_k^T Q g_k} - \frac{|g_k|^4}{g_k^T Q g_k}}{g_k^T Q^{-1} g_k} \quad (47)$$

$$= \frac{|g_k|^4}{(g_k^T Q g_k)(g_k^T Q^{-1} g_k)} \quad (\alpha_k = \frac{|g(x_k)|^2}{g(x_k)^T Q g(x_k)})$$

$$\implies q(x_k) - q(x_{k+1}) = \left(\frac{|g_k|^4}{(g_k^T Q g_k)(g_k^T Q^{-1} g_k)}\right) q(x_k) \quad (48)$$

$$\implies q(x_{k+1}) = q(x_k) \left(1 - \frac{|g_k|^4}{(g_k^T Q g_k)(g_k^T Q^{-1} g_k)}\right) \quad (49)$$

$$\leq \left(1 - \frac{4\lambda\Lambda}{(\lambda + \Lambda)^2}\right) q(x_k) \quad (\text{By Kantorovich Inequality})$$

$$\implies q(x_{k+1}) \leq \frac{(\Lambda - \lambda)^2}{(\lambda + \Lambda)^2} q(x_k) \quad (50)$$

■

Kantorovich Inequality $Q : n \times n$ positive definite symmetric matrix

$\lambda = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n = \Lambda$

For any $v \in \mathbb{R}^n$:

$$\frac{|v|^4}{(v^T Q v)(v^T Q^{-1} v)} \geq \frac{4\lambda\Lambda}{(\lambda + \Lambda)^2}$$

Theorem: Steepest Descent in Quadratic Case For any $x_0 \in \mathbb{R}^n$, method of steepest descent converges to the unique min point x_* of f .

Furthermore, for $q(x) := \frac{1}{2}(x - x_*)Q(x - x_*)$, where Q symmetric positive definite and $0 < \lambda = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n = \Lambda$,

$$q(x_{k+1}) \leq \frac{(\Lambda - \lambda)^2}{(\lambda + \Lambda)^2} q(x_k)$$

Let $r := \frac{(\Lambda - \lambda)^2}{(\lambda + \Lambda)^2}$, then

$$q(x_k) \leq r^k q(x_0)$$

for all k . As $k \rightarrow \infty$, $q(x_k) \rightarrow 0$.

Notes

1. $x_k \in \{x \in \mathbb{R}^n | q(x) \leq r^k q(x_0)\} = SL_k$
(sublevel set of function $q(x)$)

Note that SL_k is strictly decreasing. Furthermore, note that x_* is the only point satisfying the inequality at the limit:

$$q(x_*) = 0 = \lim_{k \rightarrow \infty} q(x_k)$$

Therefore, $\lim_{k \rightarrow \infty} SL_k = \{0\}$, and $x_k \rightarrow x_*$.

2. $r = (\frac{\Lambda - \lambda}{\lambda + \Lambda})^2 = (\frac{\Lambda/\lambda - 1}{\Lambda/\lambda + 1})^2$ depends only on the ratio $\frac{\Lambda}{\lambda}$ = “condition number of Q ”
case $\frac{\Lambda}{\lambda} = 1 \implies r = 0 \implies 0 \leq q(x_1) \leq 0 \cdot q(x_0) \implies q(x_1) = 0 \implies x_1 = x_*$
 (Gradient descent converges to the unique global minimum in only one iteration.)
case $\frac{\Lambda}{\lambda} \gg 1 \implies r \simeq 1$
 (worst case, converges very flow)

7.3 Method of Conjugate Direction

Motivation Method of conjugate directions is designed for quadratic functions with form $f(x) = \frac{1}{2}x^T Qx - b^T x$. For other functional forms, one can approximate the function using quadratic form firstly and then apply method of conjugate directions.

Definition: Q-orthogonality Let Q be a symmetric matrix. Two vectors $d, d' \in \mathbb{R}^n$ are Q-orthogonal (or Q-conjugate) if

$$d^T Q d' = 0$$

A finite set of d_0, \dots, d_k is called Q-orthogonal set if $d_i^T Q d_j = 0$ for all $i \neq j$.

Example 1 Q is an identity matrix. d, d' are Q -orthogonal iff they are orthogonal.

Example 2 If d, d' are two eigenvectors with different eigenvalues, then they are Q -orthogonal.

Proof. Suppose $Qv = \lambda v$ and $Qw = \lambda' w$ so $\lambda \neq \lambda'$

$$\langle v, Qw \rangle = \langle v, \lambda' w \rangle = \lambda' \langle v, w \rangle \tag{51}$$

$$= \langle Q^T v, w \rangle = \langle Qv, w \rangle = \langle \lambda v, w \rangle = \lambda \langle v, w \rangle \tag{52}$$

$$\implies (\lambda - \lambda') \langle v, w \rangle = 0 \tag{53}$$

Since $(\lambda - \lambda') \neq 0$, then we have $\langle v, w \rangle = 0$.

$$\implies v^T Q w = \langle v, Qw \rangle = \lambda \langle v, w \rangle = 0$$

■

Example 3 If Q is an $n \times n$ symmetric matrix, then there exists an orthogonal basis of eigenvectors d_0, \dots, d_{n-1}

Claim: They are also Q -orthogonal.

Proof. $d_i^T Q d_j = d_i^T (\lambda d_j) = \lambda d_i^T d_j = 0$ ■

Proposition Let Q be a symmetric positive definite matrix. Let d_0, \dots, d_k be a set of (non-zero) Q -orthogonal vectors. Then d_0, \dots, d_k are linearly independent.

Proof. Assume $\alpha_0 d_0 + \dots + \alpha_k d_k = 0$ for $\alpha_i \in \mathbb{R}$.

Multiply the whole equation by $d_i^T Q$:

$$\alpha_0 \underbrace{d_i^T Q d_0}_{=0} + \dots + \alpha_i \underbrace{d_i^T Q d_i}_{>0} + \dots + \alpha_k \underbrace{d_i^T Q d_k}_{=0} = 0$$

which implies $\alpha_i d_i^T Q d_i = 0$ and $\alpha_i = 0$.

This is true for every i . Therefore d_0, \dots, d_k are linearly independent. ■

Lemma (Theorems covered so far)

1. d_i, d_j are Q -orthogonal if $d_i^T Q d_j = 0$;
2. Eigen-vectors with different eigenvalues are Q -orthogonal;
3. Matrix Q symmetric \implies there exists an orthogonal basis \implies the set of basis is Q -orthogonal as well;
4. Q -orthogonal vectors are linearly independent.

Example 4 (Special case: Method of Conjugate Direction on Quadratic Functions). Let Q be a positive definite symmetric $n \times n$ matrix. The problem is

$$\min f(x) = \frac{1}{2} x^T Q x - b^T x$$

Recall that the unique global minimum is $x^* = Q^{-1}b$.

Let d_0, d_1, \dots, d_{n-1} be non-zero Q -orthogonal vectors.

Note that they are linearly independent by the previous theorem.

Therefore, they form a basis of \mathbb{R}^n .

The global minimum can be represented as

$$x^* = \sum_{j=0}^{n-1} \alpha_j d_j, \alpha_j \in \mathbb{R}$$

For every j , the following holds:

$$\begin{aligned} d_j^T Q x^* &= \alpha_j d_j^T Q d_j \\ \implies \alpha_j &= \frac{d_j^T Q x^*}{d_j^T Q d_j} \end{aligned}$$

Algorithm: Method of Conjugate Directions Let Q be a positive definite symmetric $n \times n$ matrix. and $\{d_j\}_{j=0}^{n-1}$ be a set of non-zero Q -orthogonal vectors, note that they form a basis of \mathbb{R}^n .

Given initial point $x_0 \in \mathbb{R}^n$, the method of conjugate direction generates a sequence of points $\{x_k\}_{k=0}^n$ as the following:

$$\begin{aligned} x_{k+1} &\leftarrow x_k + \alpha_k d_k \\ \alpha_k &:= -\frac{\langle g_k, d_k \rangle}{d_k^T Q d_k} \quad g_k := \nabla f(x_k) \end{aligned}$$

Theorem Given the method of conjugate, the sequence of points generated eventually reaches the global minimum. That is, $x_n = x^*$.

Proof. Let $x^*, x_0 \in \mathbb{R}^n$, consider

$$x^* - x_0 = \sum_{j=0}^{n-1} \beta_j d_j \quad (54)$$

$$\iff x^* = x_0 + \sum_{j=0}^{n-1} \beta_j d_j \quad (55)$$

$$d_j^T Q(x^* - x_0) = d_j^T Q\left(\sum_{j=0}^{n-1} \beta_j d_j\right) \quad (56)$$

$$= \beta_j d_j^T Q d_j \quad (57)$$

$$\implies \beta_j = \frac{d_j^T Q(x^* - x_0)}{d_j^T Q d_j} \quad (58)$$

Note that the algorithm generates the sequence as following:

$$x_k = x_0 + \sum_{j=0}^{k-1} \alpha_j d_j \quad (59)$$

$$\implies (x_k - x_0) = \sum_{j=0}^{k-1} \alpha_j d_j \quad (60)$$

$$\implies d_k^T Q(x_k - x_0) = \sum_{j=0}^{k-1} \alpha_j d_k^T Q d_j = 0 \quad (61)$$

Therefore,

$$\beta_k = \frac{d_k^T Q(x^* - x_0)}{d_k^T Q d_k} \quad (62)$$

$$= \frac{d_k^T Q(x^* - x_0) - d_k^T Q(x_k - x_0)}{d_k^T Q d_k} \quad (63)$$

$$= \frac{d_k^T Q(x^* - x_k)}{d_k^T Q d_k} \quad (64)$$

$$= \frac{d_k^T (Qx^* - Qx_k)}{d_k^T Q d_k} \quad (65)$$

$$= \frac{d_k^T (b - Qx_k)}{d_k^T Q d_k} \quad (\text{The first order necessary condition suggests } Qx^* = b)$$

$$= -\frac{d_k^T (Qx_k - b)}{d_k^T Q d_k} \quad (66)$$

$$= -\frac{d_k^T \nabla f(x_k)}{d_k^T Q d_k} \quad (\text{Assuming } f \text{ is quadratic})$$

$$= \alpha_k \quad (67)$$

Consequently,

$$x^* = x_0 + \sum_{j=0}^{n-1} \beta_j d_j \quad (68)$$

$$= x_0 + \sum_{j=0}^{n-1} \alpha_j d_j \quad (69)$$

$$= x_n \quad (70)$$

■

7.3.1 Geometric Interpretations of Method of Conjugate Directions

Theorem Let $f \in C^1(\Omega, \mathbb{R})$, where Ω is a convex subset of \mathbb{R}^n , then x_0 is a local minimum of f on Ω if and only if

$$\nabla f(x_0) \cdot (y - x_0) \geq 0 \quad \forall y \in \Omega$$

Example Now consider the special case in which Ω is an affine hyperplane, that is,

$$\Omega = \{x \in \mathbb{R}^n : cx + b = 0\}$$

where $\dim(\Omega)$ is $n - 1$.

Note that for every $y \in \Omega$, $\nabla f(x_0) \cdot (y - x_0) \geq 0$. For any feasible direction $a := y - x_0$ at point x_0 , by definition of hyperplane, $-a$ is a feasible direction as well.

Consequently, $a \cdot \nabla f(x_0) = 0$ for every feasible direction. That is, $\nabla f(x_0) \perp \Omega$. ■

Geometric Interpretation Let d_0, d_1, \dots, d_{n-1} be a set of non-zero Q -orthogonal vectors in \mathbb{R}^n . Let $B_k = \text{Span}\{d_0, \dots, d_{k-1}\}$ for $k = 0, 1, \dots, n$.

Note:

•

$$B_0 = \{0\} \subseteq B_1 = \langle d_0 \rangle \subseteq B_2 = \langle d_0, d_1 \rangle \subseteq \dots \subseteq B_n = \langle d_0, \dots, d_{n-1} \rangle = \mathbb{R}^n$$

•

$$\dim B_k = k$$

•

$$x_0 + B_0 \subseteq x_0 + B_1 \subseteq \dots$$

Theorem The sequence $\{x_k\}$ generated from $x_0 \in \mathbb{R}^n$ by conjugate directions method has the property that x_k minimizes $f(x) = \frac{1}{2}x^T Qx - b^T x$ on the affine hyperplane $x_0 + B_k$.

Proof. Recall that x_k is the minimizer of $f(x)$ on $x_0 + B_k \iff \nabla f(x_k) \perp x_0 + B_k$

Enough to prove that $\nabla f(x_k) \perp B_k$.

Remarks: x_0 here is like a bias which shifts the subspace by a “constant”. Also, “ $\nabla f(x_k) \perp B_k$ ” here means that $\nabla f(x_k)$ is perpendicular to every basis vector of B_k .

We prove this by induction on k .

Notation: $\nabla f(x_k) = Qx_k - b =: g_k$.

Base case: $k = 0$ $B_0 = \{0\} \implies g_0 \perp B_0$

Inductive Step: Assume that $g_k \perp B_k$, show $g_{k+1} \perp B_{k+1}$

Since

$$x_{k+1} = x_k + \alpha_k d_k$$

then

$$\underbrace{Q_{x_{k+1}} - b}_{g_{k+1}} = \underbrace{Q_{x_k} - b}_{g_k} + \alpha_k Q d_k$$

$$g_{k+1}^T B_k = \langle d_0, \dots, d_{k-1} \rangle \quad (71)$$

$$g_{k+1}^T d_k = \underbrace{(g_k + \alpha_k Q d_k^T d_k)^T}_{g_{k+1}} d_k \quad (72)$$

$$= g_k^T d_k + \alpha_k d_k^T Q d_k \quad (73)$$

$$= g_k^T d_k + \left(-\frac{g_k^T d_k}{d_k^T Q d_k}\right) d_k^T Q d_k \quad (74)$$

$$= 0 \quad (75)$$

This implies that $g_{k+1} \perp d_k$

For $0 \leq i < k$,

$$g_{k+1}^T \cdot d_i = (g_k + \alpha_k Q d_k)^T d_i \quad (76)$$

$$= \underbrace{g_k^T d_i}_{=0} + \underbrace{\alpha_k d_k^T Q d_i}_{=0} \quad (77)$$

$$= 0 \quad (78)$$

Therefore, $g_{k+1} \perp d_0, d_1, \dots, d_k$

Hence $g_{k+1} \perp \langle d_0, d_1, \dots, d_k \rangle = B_k$ ■

Corollary x_n minimizes $f(x)$ on $x_0 + B_n$ (which is \mathbb{R}^n)

i.e. $x_n = x^*$

Corollary $0 \leq q(x_k) = \min_{x \in x_0 + B_k} q(x) \leq q(x_{k-1}) = \min_{x \in x_0 + B_{k-1}} q(x)$

Corollary

$$\begin{cases} \min f(x) \\ x \in x_0 + B_1 \end{cases} \quad (79)$$

$$\implies \begin{cases} \min f(x_0 + t d_0) \\ t \in \mathbb{R} \end{cases} \quad (\text{Since } x_0 + B_1 = \{x_0 + t d_0 | t \in \mathbb{R}\})$$

$$\implies 0 = \frac{d}{dt} \Big|_{t=t_0} f(x_0 + t d_0) = \nabla f(x_0 + t_0 d_0) \cdot d_0 \quad (\text{where } t_0 \text{ is such that } x_1 = x_0 + t_0 d_0)$$

■

8 Calculus of Variations

Note: infinite dimensional optimization.

Comparison with finite dimensions

	finite dimensional	∞ -dimensional
problem	$\min f(x)$	$\min F[u]$
constraint	$x \in M$	$u \in \mathcal{A}$
note	set of points in \mathbb{R}^n	space of functions

Model model

$$\mathcal{A} = \{u : [0, 1] \rightarrow \mathbb{R} \mid u \in C^1 \text{ s.t. } u(0) = u(1) = 1\}$$

Note: We call F a “Functional”. It maps a function to a real number.

Notation Write $u(\cdot)$ for a function u .

8.1 Example

$$F[u(\cdot)] = \frac{1}{2} \int_0^1 \{u(x)^2 + u'(x)^2\} dx.$$

$$\begin{cases} \min F[u(\cdot)] \\ u(\cdot) \in \mathcal{A} \end{cases}$$

means: Find $u^*(\cdot) \in \mathcal{A}$ s.t. $F[u^*(\cdot)] \leq F[u(\cdot)]$ for all $u(\cdot) \in \mathcal{A}$.

Plan

1. We derive 1st order necessary conditions for a local min;
2. Find a function $u^*(\cdot)$ satisfying these conditions;
3. Check this candidate $u^*(\cdot)$ is in fact a minimizer.

We reduce this problem to (many) 1-dimensional problems.

Step 1: Derive 1st order necessary conditions for a local min

Fix $v(\cdot) \in C^1$ on $[0, 1]$ s.t. $v(0) = 0 = v(1)$.

Suppose $u^*(\cdot) \in \mathcal{A}$ is a minimizer.

Notice that $u^*(\cdot) + sv(\cdot) \in \mathcal{A} \forall s \in \mathbb{R}$.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ s.t. $f(s) := F[u^*(\cdot) + sv(\cdot)]$.

If $u^*(\cdot)$ minimizes F , then $s = 0$ minimizes f , then $f'(0) = 0$.

Then $f(0) = F[u^*(\cdot)] \leq F[u^*(\cdot) + sv(\cdot)] = f(s)$

$$f'(0) = \frac{d}{ds} \Big|_{s=0} \underbrace{F[u^*(\cdot) + sv(\cdot)]}_{f(s)} \quad (80)$$

$$= \frac{d}{ds} \Big|_{s=0} \frac{1}{2} \int_0^1 \{[u^*(x) + sv(x)]^2 + [u^{*'}(x) + sv'(x)]^2\} dx \quad (81)$$

$$= \frac{1}{2} \frac{d}{ds} \Big|_{s=0} \int_0^1 \{u^*(x)^2 + u^{*'}(x)^2\} dx + \frac{d}{ds} \Big|_{s=0} \int_0^1 \{u^*(x)v(x) + u^{*'}(x)v'(x)\} dx + \frac{d}{ds} \Big|_{s=0} \frac{s^2}{2} \int_0^1 \{v(x)^2 + v'(x)^2\} dx \quad (82)$$

$$= \int_0^1 \{u^*(x)v(x) + u^{*'}(x)v'(x)\} dx \quad (83)$$

So far, if $u^*(\cdot)$ is a minimizer of F over \mathcal{A} , then

$$\int_0^1 \{u^*(x)v(x) + u^{*'}(x)v'(x)\} dx = 0 \quad (\heartsuit)$$

for all $v(\cdot) \in C^1$ on $[0, 1]$ and $v(0) = 0 = v(1)$. We call this a “primitive form of 1st order condition”, and call $v(\cdot)$ the test functions.

Recall Integration by parts:

$$\int_0^1 w(x)v'(x) dx = w(x)v(x)|_0^1 - \int_0^1 w'(x)v(x) dx$$

$$(\heartsuit) = \int_0^1 u^*(x)v(x) dx + \int_0^1 u'^*(x)v'(x) dx \quad (84)$$

$$= \int_0^1 u^*(x)v(x) dx + \underbrace{u'^*(x)v(x)|_0^1}_{=0 \text{ (} v(0)=v(1)=0 \text{)}} - \int_0^1 u''^*(x)v(x) dx \quad (85)$$

$$= \int_0^1 (u^*(x) - u''^*(x)) v(x) dx \quad (86)$$

$$= 0 \quad (87)$$

For all test functions $v(\cdot)$.

Lemma: Fundamental Lemma of Calculus of Variations Suppose g is continuous function on interval $[a, b]$. If

$$\int_a^b g(x)v(x) dx = 0$$

for all test functions $v(\cdot)$, then

$g(x) \equiv 0$ on $[a, b]$.

Then by Fundamental Lemma of Calculus of Variations, $(\heartsuit) \implies u^*(x) - u''^*(x) \equiv 0$, which is the 1st order necessary condition for $u^*(\cdot)$.

Step 2: Find a function $u^*(\cdot)$ satisfying these conditions

$$\begin{cases} u^*(x) = u''^*(x) \\ u^*(0) = u^*(1) = 1 \end{cases} \implies u^*(x) = c_1 e^x + c_2 e^{-x} \quad (88)$$

$$\begin{cases} 1 = u^*(0) = c_1 + c_2 \\ 1 = u^*(1) = c_1 e + c_2 \frac{1}{e} \end{cases} \implies c_1 = \frac{1}{e+1}, c_2 = \frac{e}{e+1} \quad (89)$$

$$\implies u^*(x) = \frac{1}{e+1} e^x + \frac{e}{e+1} e^{-x} \quad (90)$$

Step 3: check $u^*(\cdot)$ is in fact a global minimizer.

We derived that

$$F[u^*(\cdot) + sv(\cdot)] = F[u^*(\cdot)] + s \underbrace{\int_0^1 \{u^*(x)v(x) + u'^*(x)v'(x)\} dx}_{=0} + \underbrace{\frac{s^2}{2} \int_0^1 \{v(x)^2 + v'(x)^2\} dx}_{\geq 0}$$

$$F[u^*(\cdot)] \leq F[u^*(\cdot) + sv(\cdot)]$$

for all test functions $v(\cdot)$ and all $s \in \mathbb{R}$. In particular, let $s = 1$, then

$$F[u^*(\cdot)] \leq F[u^*(x) + v(\cdot)]$$

for all test functions $v(\cdot)$. In particular, let $v(\cdot) = u(\cdot) - u^*(\cdot)$, then

$$F[u^*(\cdot)] \leq F[u(\cdot)]$$

for all $u(\cdot) \in \mathcal{A}$.

Note: The space of $v(\cdot)$ is a vector space, but \mathcal{A} is not a vector space (since $u(\cdot) \neq 0$). It is a translate of a vector space.

Lemma: Fundamental Lemma of Calculus of Variations Suppose g is continuous function on interval $[a, b]$. If

$$\int_a^b g(x)v(x) dx = 0$$

for all test functions $v(\cdot)$, then
 $g(x) \equiv 0$ on $[a, b]$.

Proof. Suppose for contradiction that $g(x) \not\equiv 0$ on $[a, b]$.

WLOG, $g(x_0) > 0$ for some $x_0 \in (a, b)$. This implies that $g > 0$ on $(c, d) \subsetneq (a, b)$.

Let $v(\cdot)$ be a continuous function s.t.

$$v(\cdot) \begin{cases} > 0 & \text{on } (c, d) \\ = 0 & \text{otherwise} \end{cases}$$

Then

$$\int_a^b g(x)v(x) dx = \int_c^d \underbrace{g(x)v(x)}_{>0} dx > 0$$

which leads to a contradiction. ■

8.2 Classical Problem: the Brachistochrone

Galileo (1638): Find the curve connecting A and B on which a point mass moves without friction under the influence of gravity in the least time possible.

Johann Bernoulli (1696): Revisit the problem and sent invitations

6 correct solutions sent (1697):

Leibniz, Johann, [Jacob](#), l'Hospital, Von Tschinhaus, Anonymous \rightarrow Newton (*)

[This answer is the beginning of Calculus of Variations](#)

8.3 General class of problems in Calculus of Variations

$$\mathcal{A} = \{u : [a, b] \rightarrow \mathbb{R} \mid u \in C^1, u(a) = A, u(b) = B\}$$

$$F[u(\cdot)] = \int_a^b L(x, u(x), u'(x)) dx$$

where $L(x, z, p) : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$

Model example $L(x, z, p) = \frac{z^2 + p^2}{2}, \quad F[u(\cdot)] = \int_0^1 \frac{u(x)^2 + u'(x)^2}{2} dx$

Notation:

$$\begin{cases} L_z(x, z, p) = \frac{\partial}{\partial z} L(x, z, p) \\ L_p(x, z, p) = \frac{\partial}{\partial p} L(x, z, p) \end{cases}$$

Definition Given $u(\cdot) \in \mathcal{A}$, suppose \exists function $g(\cdot)$ on $[a, b]$ s.t.

$$\frac{d}{ds} \Big|_{s=0} F[u(\cdot) + sv(\cdot)] = \int_a^b g(x)v(x) dx$$

for all test functions $v(\cdot)$, then $g(\cdot)$ is called the variational derivative of F at $u(\cdot)$, denoted by $\frac{\delta F}{\delta u}(u)(\cdot)$ or $\frac{\delta F}{\delta u}(u)$ or $\frac{\delta F}{\delta u}$.

Analogy In finite dimensions:

$$\frac{d}{ds}\big|_{s=0} f(u + sv) = \nabla f(u) \cdot v \quad (91)$$

$$= \sum_i \nabla f(u)_i v_i \quad (92)$$

for all $v \in \mathbb{R}^n$

In Calculus of Variations (∞ dimensional):

$$\frac{d}{ds}\big|_{s=0} F[u(\cdot) + sv(\cdot)] = \int_a^b \frac{\delta F}{\delta u}(u)(x) v(x) dx \quad (\text{where possible})$$

$$\sim \sum_x \frac{\delta F(u)}{\delta u}(x) v(x) \quad (\text{a kind of an infinite sum})$$

Model example $L(x, z, p) = \frac{z^2 + p^2}{2}, \quad F[u(\cdot)] = \int_0^1 \frac{u(x)^2 + u'(x)^2}{2} dx$

$$\frac{d}{ds}\big|_{s=0} F[u(\cdot) + sv(\cdot)] = \dots = \int_0^1 [u(x) - u''(x)] v(x) dx$$

for all test functions $v(\cdot)$

Therefore $\frac{\delta F}{\delta u}(u)(x) = u(x) - u''(x)$

Lemma (1st order necessary conditions satisfied by a solution $u^*(\cdot) \in C^1$)

$$\mathcal{A} = \{u : [a, b] \rightarrow \mathbb{R} \mid u \in C^1, u(a) = A, u(b) = B\}$$

If $u^*(\cdot) \in \mathcal{A}$ minimizes F over \mathcal{A} , and if $\frac{\delta F}{\delta u}(u^*)(\cdot)$ exists and is continuous, then it must satisfy

$$\frac{\delta F}{\delta u}(u^*)(\cdot) \equiv 0$$

Proof. note: $u^*(\cdot) + sv(\cdot) \in \mathcal{A}$

If $u^*(\cdot)$ is a minimizer of F , then

$$F[u^*(\cdot)] \leq F[u^*(\cdot) + sv(\cdot)]$$

for all test functions v .

Define $f(s) := F[u^*(\cdot) + sv(\cdot)]$, then $f(0) \leq f(s)$ for all $s \in \mathbb{R}$.

Then

$$\int_a^b \frac{\delta F}{\delta u}(u^*)(\cdot) v(x) dx = \frac{d}{ds}\big|_{s=0} F[u^*(\cdot) + sv(\cdot)] \quad \left(\frac{\partial F}{\partial u}(u^*)(\cdot) \text{ exists}\right)$$

$$= \frac{d}{ds}\big|_{s=0} f(s) \quad (93)$$

$$= f'(0) = 0 \quad (0 \text{ is the global minimize of } f)$$

This implies that $\frac{\delta F}{\delta u}(u^*)(\cdot) \equiv 0$. ■

Theorem: Leibniz Integral Rule Let $f(x, t)$ be a function such that both $f(x, t)$ and its partial derivative $\frac{\partial}{\partial x} f(x, t)$ is continuous w.r.t. t and x in some region of the (x, t) -plane, including $a(x) \leq t \leq b(x), x_0 \leq x \leq x_1$. Also suppose that the functions $a(x)$ and $b(x)$ are both continuous and both have continuous derivatives for $x_0 \leq x \leq x_1$. Then, for $x_0 \leq x \leq x_1$,

$$\frac{d}{dx} \left(\int_{a(x)}^{b(x)} f(x, t) dt \right) = f(x, b(x)) \cdot \frac{d}{dx} b(x) - f(x, a(x)) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt$$

Note that if $a(x)$ and $b(x)$ are constants rather than functions of x , we have a special case of Leibniz's rule:

$$\frac{d}{dx} \left(\int_a^b f(x, t) dt \right) = \int_a^b \frac{\partial}{\partial x} f(x, t) dt$$

Theorem

$$\mathcal{A} = \{u : [a, b] \rightarrow \mathbb{R} | u \in C^1, u(a) = A, u(b) = B\}$$

$$F[u(\cdot)] = \int_a^b L(x, u(x), u'(x)) dx$$

where $L \in C^2$.

Then if $u(\cdot) \in C^1$, then $\frac{\delta F}{\delta u}(u)(\cdot)$ exists, is continuous, and

$$\frac{\delta F}{\delta u}(u)(x) = -\frac{d}{dx}[L_p(x, u(x), u'(x))] + L_z(x, u(x), u'(x))$$

Let v be a test function ($v(a) = v(b) = 0$)

$$\frac{d}{ds}|_{s=0} F[u(\cdot) + sv(\cdot)] = \frac{d}{ds}|_{s=0} \int_a^b L(x, u(x) + sv(x), u'(x) + sv'(x)) dx \quad (94)$$

$$= \int_a^b \frac{d}{ds}|_{s=0} L(x, z, p) dx \quad (\text{By Leibniz's rule})$$

$$= \int_a^b L_z(\cdot)v(x) + L_p(\cdot)v'(x) dx \quad (95)$$

$$= \int_a^b L_z(\cdot)v(x) dx + \int_a^b L_p(\cdot)v'(x) dx \quad (96)$$

$$= \int_a^b L_z(\cdot)v(x) dx + \textcolor{red}{L_p(\cdot)v(x)}|_a^b - \int_a^b \frac{d}{dx} L_p(\cdot)v(x) dx \quad (\text{Integration by parts})$$

$$= \int_a^b [-\frac{d}{dx} L_p(\cdot) + L_z(\cdot)]v(x) dx \quad \forall \text{ test functions } v(\cdot) \quad (97)$$

By the definition of variational derivative, it follows that

$$\frac{\delta F}{\delta u}(u)(x) = -\frac{d}{dx}[L_p(x, u(x), u'(x))] + L_z(x, u(x), u'(x))$$

Furthermore, since $L(\cdot) \in C^2$, $-\frac{d}{dx} L_p(\cdot)$ and $L_z(\cdot)$ are continuous. Moreover, $u(\cdot)$ and $u'(\cdot)$ are continuous, so is the composite function. Hence the variational derivative is continuous.

Model example $L(x, z, p) = \frac{z^2 + p^2}{2}$, $F[u(\cdot)] = \int_0^1 \frac{u(x)^2 + u'(x)^2}{2} dx$

$$L_z(x, z, p) = z \implies L_z(x, u(x), u'(x)) = u(x)$$

$$L_p(x, z, p) = p \implies L_p(x, u(x), u'(x)) = u'(x)$$

$$\frac{\delta F}{\delta u}(u(\cdot)) = -\frac{d}{dx}[u'(x)] + u(x) = -u''(x) + u(x)$$

If $u^*(\cdot) \in \mathcal{A}$ is a minimizer, then $-u''(x) + u(x) = 0$

Example 8.1 (min arclength). We will show that the straight line gives the shortest path.

$$\min F[u(\cdot)] = \int_a^b (1 + u'(x)^2)^{\frac{1}{2}} dx = \text{arclength of } u(\cdot)$$

$$\mathcal{A} = \{u : [a, b] \rightarrow \mathbb{R} | u \in C^1, u(a) = A, u(b) = B\}$$

Then $L(x, z, p) = (1 + p^2)^{\frac{1}{2}}$, $L_z = 0$ and $L_p = \frac{p}{(1 + p^2)^{\frac{1}{2}}}$

If $u^*(\cdot)$ is a minimizer, then

$$-\frac{d}{dx} \frac{u'(x)}{(1 + u'(x)^2)^{\frac{1}{2}}} \equiv 0$$

$\implies \frac{u'(x)}{(1+u'(x))^{\frac{1}{2}}} = \text{const}$
 $\implies u'(x)^2 = \text{const}(1+u'(x)^2)$
 Then $u'(x) = \alpha$ for some $\alpha \in \mathbb{R}$.
 Then $u(x) = \alpha x + \beta$ for some $\beta \in \mathbb{R}$.

Example 8.2 (Surface Area of Revolution). Suppose $u(\cdot) \in C^1$ on $[a, b]$, the surface area of rotating the curve u connecting a and b can be computed as

$$F[u(\cdot)] = \int_a^b 2\pi u(x) \sqrt{1 + u'(x)} dx$$

For simplicity, assume $u > 0$. In this example, the space of feasible functions is

$$\mathcal{A} = \{u : [a, b] \rightarrow \mathbb{R} : u \in C^1, u(a) = A, u(b) = B, u > 0\}$$

If $u(\cdot)$ solves the minimization problem, it must be the case that

$$\frac{\delta F}{\delta u}(u)(\cdot) \equiv 0 \quad (\dagger)$$

Notice

$$\begin{aligned}
 L(x, z, p) &= 2\pi z \sqrt{1 + p^2} \\
 L_z(x, z, p) &= 2\pi \sqrt{1 + p^2} \\
 L_p(x, z, p) &= 2\pi z \frac{p}{\sqrt{1 + p^2}}
 \end{aligned}$$

Claim: the family of $u(\cdot) = \beta \cosh(\frac{x-\alpha}{\beta})$ solves the necessary condition \dagger .

Instance 1 When $a = 0, b = 1, A = B = 1$, plugging in the initial condition gives

$$\begin{cases} \beta \cosh\left(\frac{0-\alpha}{\beta}\right) &= 1 \\ \beta \cosh\left(\frac{1-\alpha}{\beta}\right) &= 1 \end{cases} \quad (98)$$

solving above system of equations provides the solution.

Instance 2 When $a = 0, b = 1, A = 1, B = 0$, plugging in these initial conditions gives

$$\begin{cases} \beta \cosh\left(\frac{0-\alpha}{\beta}\right) &= 1 \\ \beta \cosh\left(\frac{1-\alpha}{\beta}\right) &= 0 \end{cases} \quad (99)$$

because $\cosh > 0$, the second equation suggests $\beta = 0$, but in this case the first equation would never hold. Therefore, there is no solution to this calculus of variation.

In face, the surface area is minimized by

$$u(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \quad (100)$$

8.4 Euler-Lagrange Equations in \mathbb{R}^n

Setup

$$F[u(\cdot)] = \int_a^b L(x, u(x), u'(x)) dx \quad (101)$$

$$u : [a, b] \rightarrow \mathbb{R}^n \quad (102)$$

$$L(x, z, p) : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \quad (103)$$

$$\mathcal{A} := \{u : [a, b] \rightarrow \mathbb{R}^n : u \in C^1, u(a) = \mathbf{A}, u(b) = \mathbf{B}\} \quad (104)$$

Theorem 8.1 (Euler-Lagrange Equations in Vector Forms).

$$-\frac{d}{dx}\nabla_p L(x, z, p) + \nabla_z L(x, z, p) = \mathbf{0} \in \mathbb{R}^n \quad (\dagger) \quad (105)$$

Example 8.3 (Classical Lagrangian Mechanics).

$$V(x) : \mathbb{R}^n \rightarrow \mathbb{R} \text{ potential energy} \quad (106)$$

$$\frac{1}{2}m\|v\|_2^2 \text{ kinetic energy} \quad (107)$$

$$L(t, x, v) := \frac{1}{2}m\|v\|_2^2 - V(x) \text{ difference between KE and PE} \quad (108)$$

Consider a path $x(t)$ in \mathbb{R}^n , define objective function as

$$F[x(\cdot)] = \int_a^b L(t, x(t), x'(t)) dt \quad (109)$$

$$= \int_a^b \frac{1}{2}m\|\dot{x}(t)\|_2^2 - V(x(t)) dt \quad (110)$$

The Euler-Lagrange equation in vector form implies

$$-\frac{d}{dt}\nabla_{(3)}L(t, x(t), \dot{x}(t)) + \nabla_{(2)}L(t, x(t), \dot{x}(t)) = 0 \quad (111)$$

$$\implies -\frac{d}{dt}m\dot{x}(t) - \nabla V(x(t)) = 0 \quad (112)$$

$$\implies m\ddot{x}(t) = \nabla V(x(t)) \quad (\dagger\dagger) \quad (113)$$

Remark 8.1. $(\dagger\dagger)$ is often referred to as *Newton's second law*: object moves along the path on which the total conversion between kinetic and potential energies is minimized.

Example 8.4 (3-Dimensional Pendulum). Suppose the pendulum is moving on a path such that the total conversion between kinetic and potential energies is minimized, that is

$$\min \int_a^b L(t) dt = \int_a^b \frac{1}{2}m(\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2) - mgz(t) dt \quad (114)$$

with the restriction that $\|\mathbf{x}(t)\| = \ell$, where ℓ is the radius of the sphere.

The restriction can be embodied by framing the problem using spherical coordinates:

$$x := \ell \cos \varphi \sin \theta \quad (115)$$

$$y := \ell \sin \varphi \sin \theta \quad (116)$$

$$z := -\ell \cos \theta \quad (117)$$

where the path of motion can be characterized using $(\theta(t), \varphi(t))$.

The objective function is therefore

$$L\left(t, \begin{pmatrix} \theta(t) \\ \varphi(t) \end{pmatrix}, \begin{pmatrix} \dot{\theta}(t) \\ \dot{\varphi}(t) \end{pmatrix}\right) = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2(\theta)) + mg\ell \cos \theta \quad (118)$$

So the Euler-Lagrange equation can be written as

$$-\frac{d}{dt}\nabla_{(3)}L\left(t, \begin{pmatrix} \theta(t) \\ \varphi(t) \end{pmatrix}, \begin{pmatrix} \dot{\theta}(t) \\ \dot{\varphi}(t) \end{pmatrix}\right) + \nabla_{(2)}L\left(t, \begin{pmatrix} \theta(t) \\ \varphi(t) \end{pmatrix}, \begin{pmatrix} \dot{\theta}(t) \\ \dot{\varphi}(t) \end{pmatrix}\right) = \mathbf{0} \quad (119)$$

8.5 Equality constraints

8.5.1 Isoperimetric constraints

Recall: finite dimensional case

$$f, g : \mathbb{R}^n \rightarrow \mathbb{R} \quad (120)$$

$$\begin{cases} \min_{x \in \mathbb{R}^n} & f(x) \\ & g(x) = \text{const} \end{cases} \quad (121)$$

Suppose regular point x_* ($\nabla g(x_*) \neq 0$) is a minimizer. Then $\exists \lambda \in \mathbb{R}$ s.t. $\nabla f(x_*) + \lambda \nabla g(x_*) = 0$ (by Lagrange multipliers)

Remark 8.2. x_* minimizes $f + \lambda g$. The Lagrange multipliers convert the original problem to an unconstrained optimization problem $L(x, \lambda) = f(x) + \lambda g(x)$.

Infinite dimensional case

$$F[u(\cdot)] = \int_a^b L^F(x, u(x), u'(x)) dx \quad (122)$$

$$G[u(\cdot)] = \int_a^b L^G(x, u(x), u'(x)) dx \quad (123)$$

$$(124)$$

$$\begin{cases} \min_{u(\cdot) \in \mathcal{A}} & F[u(\cdot)] \\ & G[u(\cdot)] = \text{const} \end{cases}$$

Suppose regular point $u_*(\cdot)$ ($\frac{\delta G}{\delta u}(u_*) \neq 0$) is a minimizer, then $\exists \lambda \in \mathbb{R}$ s.t.

$$\frac{\delta F}{\delta u}(u_*) + \lambda \frac{\delta G}{\delta u}(u_*) \equiv 0$$

Remark 8.3. u_* minimizes $F + \lambda G$.

Example 8.5.

$$\mathcal{A} := \{u : [-a, a] \rightarrow \mathbb{R}, u \in C^1, u(-a) = u(a) = 0\} \quad (125)$$

$$F[u(\cdot)] = \int_a^b u(x) dx \quad (126)$$

$$G[u(\cdot)] = \int_a^b \sqrt{1 + u'(x)} dx = l > 0 \quad \text{note that } G \text{ is arc length} \quad (127)$$

$$\begin{cases} \min_{u \in \mathcal{A}} & (-F)[u(\cdot)] \\ & G[u(\cdot)] = l \end{cases} \quad (128)$$

Let $u_*(\cdot)$ be a minimizer, then

$$\begin{aligned} \frac{\delta F}{\delta u} &= -\frac{d}{dx} L_p^F + L_z^F \\ \frac{\delta G}{\delta u} &= -\frac{d}{dx} L_p^G + L_z^G \end{aligned}$$

Then Euler-Lagrange equations suggests

$$\begin{aligned} -\frac{d}{dx} L_p^F + L_z^F + \lambda \left(-\frac{d}{dx} L_p^G + L_z^G \right) &= 0 \\ \implies \lambda^2 \frac{u'_*(x)^2}{1 + u'_*(x)^2} &= (C_1 - x)^2 (\dagger) \end{aligned}$$

Claim: Solution $u_*(\cdot)$ to (†) satisfies

$$(x - C_1)^2 + (u_*(x) - C_2)^2 = \lambda^2$$

So that the graph of $u_*(\cdot)$ lies on a circle, and our solution is the semi-circle that has length l .
Check:

$$\frac{d}{dx} [2(x - C_1) + 2(u_*(x) - C_2)u'_*(x)] = 0 \quad (129)$$

which implies

$$u'_*(x) = -\frac{x - C_1}{u_*(x) - C_2} \quad (130)$$

$$\implies u'_*(x)^2 = \frac{(x - C_1)^2}{(u_*(x) - C_2)^2} \quad (\S) \quad (131)$$

Also,

$$(u'_*(x)^2)(u_*(x) - C_2)^2 = (x - C_1)^2 + (u_*(x) - C_2)^2 = \lambda^2 \quad (132)$$

$$\implies (u_*(x) - C_2)^2 = \frac{\lambda^2}{1 + u'_*(x)^2} \quad (\S\S) \quad (133)$$

Combine (§) and (§§),

$$\frac{\lambda^2}{1 + u'_*(x)^2} u'_*(x)^2 = (x - C_1)^2 \quad (134)$$

It is possible to solve for u :

$$\begin{cases} x = -a, y = 0, (-a - C_1)^2 + (0 - C_2)^2 = \lambda^2 \\ x = +a, y = 0, (+a - C_1)^2 + (0 - C_2)^2 = \lambda^2 \\ \int_{-a}^a \sqrt{1 + u'(x)^2} dx = l \end{cases}$$

8.5.2 Holonomic constraints

Setup(3-Dim Special Case) Minimize

$$F[x(\cdot), y(\cdot), z(\cdot)] = \int_a^b L(t, x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t)) dt$$

with constraint

$$H(x(t), y(t), z(t)) \equiv 0$$

Theorem 8.2 (Euler-Lagrange Equations). Let $\mathbf{x}_*(t) := \begin{pmatrix} x_*(t) \\ y_*(t) \\ z_*(t) \end{pmatrix}$ be the minimizer subject to the constraint,

then

$$\begin{pmatrix} \frac{\delta F}{\delta x} [x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \\ \frac{\delta F}{\delta y} [x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \\ \frac{\delta F}{\delta z} [x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \end{pmatrix} + \lambda(t) \begin{pmatrix} H_x[x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \\ H_y[x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \\ H_z[x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \end{pmatrix} = 0 \quad \forall t \in \mathbb{R}$$

where $\lambda : [a, b] \rightarrow \mathbb{R}$ is a function.